Stochastic Dominance, Stochastic Volatility and the Prices of Volatility and Jump Risk

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Abstract

We show that the stochastic dominance (SD) approach to the valuation of index options in frictionless markets allows the derivation of a unique variance risk premium and price of volatility risk based only on the underlying return and volatility dynamics for a wide class of stochastic volatility (SV) models. The SD approach also derives under similar conditions tight bounds on the admissible option prices in frictionless markets when there are independent jumps together with SV, the SVJ models. We demonstrate numerically by using published results from high profile studies the differences that our volatility risk prices yield in option values in comparison to prices extracted from observed option market data. We also present strategies that allow the profitable exploitation of the differences between model values and observed option market data, and out-of-sample tests of the ex post profitability of such strategies in the frictionless world.

Keywords: Option Pricing, Stochastic Dominance, Stochastic Volatility, Pricing kernel, Jumps in Returns

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I. Introduction

Stochastic volatility (SV) models appeared relatively early in the valuation of derivative securities, before Finance academics and professionals recognized that the seminal Black-Scholes-Merton (BSM, 1973) model, based on constant volatility diffusion dynamics for the evolution of a derivative-underlying security, was no longer working. Indeed, in the year 1987 at least four journal articles appeared in the literature, presenting SV models that attempted to relax the constant volatility of BSM.[2] Surprisingly, this outburst of interest in extending the BSM took place soon after an important empirical article by Rubinstein (1985) had concluded that BSM was an adequate option valuation tool. Nonetheless, the SV studies were prescient, since 1987 was also the year of the major stock market crash that effectively signified the end of BSM. Indeed, in his 1994 presidential address Rubinstein reported that assigning a common volatility to the observed options prices of S&P 500 index options of a given maturity produced errors of a magnitude that made him conclude that the constant volatility BSM was no longer an approximately acceptable option model. Since that time SV models have become standard components of empirical index option research, often combined with independent Poisson jumps, the SVJ models. The list of references is very large, and will be reviewed further on in this section.

What has not changed very much during this long time interval is the transition from the index return distribution based on parameter estimates extracted from the observed returns (the physical or P-parameters), to the distribution needed for the pricing of derivatives, the risk neutral or Q-distribution. Asset pricing theory stipulates that such a transition must satisfy a set of relations that reflect the simultaneous no arbitrage equilibrium (NAE) of the index and option markets, the dominant paradigm in empirical options research. This equilibrium involves the pricing kernel, identified with the aggregation of the marginal utilities of the traders in these two markets. If the kernel is known the equilibrium relations yield the Q-distribution from the estimated P-parameters. Since the kernel is not known, the Q-parameters are extracted from the observed option market prices. This extraction involves two assumptions, that the option equilibrium prices are unique and observable (a frictionless market) and that the option equilibrium is efficient, in the sense that the observed prices are "correct". Both assumptions are questionable, and their relaxation forms the central topic of this paper, in which the transition from the P- to the Q-distribution does not involve option market data.

We find that this transition is uniquely defined when we use the frictionless stochastic dominance (SD) approach for a very wide class of SV models that covers all those that have appeared in practice. Similarly, we find that in combining SV with an independent jump process and transforming it into an SVJ model of ex-dividend index return dynamics the transition from the P- to the Q-distribution defines two limiting distributions that contain all SD-admissible option values. These novel theoretical results allow us to assess whether any frictionless equilibrium consistent with the estimated parameters of the P-process can be extracted from the observed option market data. Although there are no empirical applications in this paper, we use

[2] See Hull and White (1987), Johnson and Shanno (1987), Scott (1987), and Wiggins (1987).

P-parameters from existing studies to illustrate the difference in option values between our SD approach and the empirically fitted kernels to observed option market values. Note that the only additional assumption of SD vis-a-vis NAE is the monotonicity of the pricing kernel, which is discussed further on in this section.

SD appeared at about the same time as the SV studies, in a series of articles that started with Perrakis and Ryan (1984), Levy, (1985), Ritchken (1985), Perrakis (1986) and Ritchken and Kuo (1988). These studies adopted a discrete time methodology, unlike SV, even though they assumed a frictionless option market. The SD approach was subsequently extended to option markets with frictions, theoretically by Constantinides and Perrakis (2002, 2007) and empirically by Constantinides, Jackwerth and Perrakis (2009), Constantinides *et al* (2011), Constantinides, Czerwonko and Perrakis (2020), Post and Longarela (2021), and Arvanitis, Post and Topaloglou (2021). Strikingly, these empirical studies are to our knowledge the only empirical studies in index option markets that recognize the option bid and ask prices and do not manipulate the data by imposing a frictionless market format.

On the other hand, SD has not had any impact on option research in the frictionless world, where most of the empirical studies on index options have taken place. It was relatively recently extended to continuous time constant volatility and state-dependent volatility diffusion, and constant volatility jump diffusion in models without frictions. It forms, therefore, an alternative paradigm to NAE for option valuation both in the frictionless world and in the presence of frictions for these types of P-distributions.[3] This paper, by extending SD frictionless option pricing to SV and SVJ P-dynamics, completes this paradigm and opens up new avenues for empirical option research. Our setting is quite flexible and general and admits an ex-dividend risk premium for the index that may be constant, proportional to volatility, proportional to variance, or any other function of volatility. It also recognizes several well-known empirical features of index option research such as the leverage effect, and allows when combined with jumps different risk premiums for jumps depending on option maturity. In other words, the SD approach is an efficient method to achieve dynamically complete frictionless option markets under SV, as well as tight option bounds under general conditions for SVJ.

To our knowledge, such an endogenous derivation has never been done theoretically in NAE models under such general conditions. In the Heston (1993) SV model the kernel was derived twenty years after it first appeared only under a risk premium that is linear in the variance, by Christoffersen, Heston and Jacobs (2013).[4] Attempts to derive the Q-distribution for the SV model in the mathematical finance literature by Romano and Touzi (1997) and Frey and Sin (1999) were not particularly successful. The first one used the option market for this purpose while the latter derived option bounds that were based on arbitrarily set bounds on volatility and have never, to our knowledge been used empirically. Similarly restrictive were stylized equilibrium models in an economy in which the driving component was the optimal portfolio selection of a representative investor who maximized the discounted additive expected utility of the constant relative risk aversion (CRRA) type. This was done by Bates (1991) for options with

[3] See Perrakis (2019) and Ghanbari, Oancea and Perrakis (2021).

[4] As Jones points out (2003, p. 181), the Heston model "is incapable of generating realistic returns behavior".

constant volatility jump diffusion, and by Amin and Ng (1993) for options separately under SV and under constant volatility jump diffusion. Unlike SD, the derived results in these two studies are not preference-free, insofar as the same relative risk aversion (RRA) of the representative investor enters into the expressions that yield the Q-parameters as functions of the P-parameters.

The fact that the same RRA value is used for all maturities and degrees of moneyness of the options is problematic, since it is contradicted by one of the earliest empirical post-BSM studies on S&P 500 index options, by Bakshi, Cao and Chen (BCC, 1997). In that study various nested models of Q-dynamics were fitted to option market data, disaggregated by categories of maturity length and degree of moneyness. The results of that much-cited empirical study point very clearly to a maturity effect in the consistency of the estimates with each other and the ability to project out-of-sample option values. As the authors state (p. 214), "short term options…present perhaps the greatest challenge to any alternative [to the BSM] option pricing model". Further, the authors point out (p. 2029) that their estimates also imply a degree of moneyness effect, in the sense that all of their models systematically overprice OTM calls while they systematically underprice in-the-money (ITM) calls, with respect to the observed out-of-sample prices of these same options one day ahead. These maturity and moneyness effects imply that any risk neutralization based on a NAE model with a CRRA representative investor is not appropriate for index option valuation.

These empirical problems of the models used to represent the index options have not disappeared in the twenty-four years since the Bakshi, Cao and Chen study was published, something that is perhaps not surprising since the main features of the models and the structure of the empirical work have changed very little during that time. All of them conform to the NAE framework. The main methodological change vis-à-vis the BCC study has been the simultaneous estimation of the P- and Q-distributions, respectively from the underlying and the option markets, generally assumed of a similar form for mathematical tractability. Studies that have applied the SVJ model to S&P 500 options include, in addition to those already mentioned, Andersen, Fusari and Todorov (2017), Bates (1996, 2003, 2006), Bondarenko (2003, 2014), Broadie, Chernov and Johannes (2007, 2009), Chen, Joslin and Ni (2019), Eraker (2004), Eraker, Johannes and Polson (2003), Pan (2002) and Ziegler (2007).[5] Another category of empirical studies starting with Jackwerth and Rubinstein (1996) and Jackwerth (2000) extracts the pricing kernel as the ratio of the Q- to P-densities, by fitting these densities numerically and non-parametrically to the observed index returns and the option midpoints without assuming specific forms for the dynamics.

The general flavor of these empirical studies can be judged by the fact that several of them have been devoted to debating two related subjects, on which the empirical evidence is contradictory and contentious, both of which appeared for the first time in the aforementioned Jackwerth

[5] A parallel set of empirical studies on these same options have replaced SV with discrete-time GARCH (General Autoregressive Conditional Heteroscedasticity) models, also combined with a jump process. These are also amenable to an SD formulation but present different challenges from SV and will not be examined in this paper.

(2000) study. The first one is that OTM put options have been "too expensive", in the sense that adopting short positions in them is a highly profitable strategy. The second and much more important controversial Jackwerth finding is that the shape of the kernel, which was monotone decreasing in the index value, changed shape and became at times increasing after the 1987 crash. Empirical studies since then have confirmed or refuted both Jackwerth results, especially the case of the possibly non-monotonicity of the kernel, which was confirmed and refuted in two studies that appeared simultaneously in 2018 and refuted again in another study that appeared in 2020.[6] These contradictory studies' results for both OTM put overpricing and kernel monotonicity were analyzed and summarized in Perrakis (2022) and will not be repeated here. It suffices to state that these results came out of the same data and the same general NAE class of models and differed only in the handling of the option data and the estimation methodology. They are the best justification for the SD approach, to which we now turn.

The fact that SD can achieve results in solving a problem that has persisted for so many years in the dominant NAE approach of valuing derivative securities is due to its differing set of assumptions. As argued in Perrakis (2022), NAE is equivalent to first order stochastic dominance in our setup, while in all its applications to derivatives pricing SD is second order dominance or higher. Indeed, in several of the derivations of the BSM model the derivatives valuation methodology has used either the replication of the derivative with the underlying and a riskless security, or the construction of a continuously rebalanced perfectly hedged portfolio containing these two securities plus the option. Violations of this replicating value imply that a long or short position in the derivative yields a first degree dominance over the replicating portfolio. These replication techniques are, however, unable to yield results when the P-dynamics go beyond simple diffusion, even in frictionless markets.[7]

In frictionless SD models the basic assumption is that a generic risk averse investor holds an optimally selected portfolio of the index and a riskless bond and adds a marginal position in a single call or put option. The option, whether short or long, should not create second degree stochastically dominant positions, implying that its price should lie within bounds. These bounds depend on the probability distribution of the ex-dividend index returns, which is taken as given, but are otherwise model free. The investor portfolio composition assumption appears to be restrictive, unless the index is assumed to be the market portfolio.[8] This is also what was assumed in the aforementioned studies of Bates (1991) and Amin and Ng (1993), with the important difference that the expected ex-dividend return of the index is determined simultaneously together with the price of the option in the portfolio choices of a CRRA investor.

The SD method is formulated in discrete time, but the bounds can be applied recursively for any horizon length till option maturity. If applied to the Euler discretization of the continuous time index returns it can derive the corresponding Q-distributions by letting the time partition go to 0.

[6] See Babaoglu *et al* (2018), Linn, Shive and Shumway (2018), and Barone-Adesi *et al* (2020).
[7] They are also unable to accommodate proportional transaction costs in trading the underlying, as shown in a long list of references surveyed in detail in Perrakis, (2019, pp. 90-96).
[8] The assumption is also justified by the large and drastically increasing share of index-holding investors, as documented by Bogle (2005) and Charles (2017).

This exercise was carried out in Perrakis (2019, Chapter 2), and it was shown that the dependence on the index risk premium disappears under constant volatility and state-dependent volatility diffusion, for which the two bounds converge respectively to the BSM option price and the Constant Elasticity of Variance (CEV) price. By contrast, in the presence of jumps the bounds at the limit do not depend on the premium but remain distinct and generate an interval of admissible option prices.

The main advantage of frictionless SD vis-à-vis NAE is the fact that it derives option values or bounds for these values that do not require option market data. The option market is an intermediated market, in which there are two distinct sets of traders, market makers or dealers and end users. To our knowledge, this market has never been modeled except in a trivial sense, by Garleanu, Petersen and Poteshman (2009) and more recently by Fournier and Jacobs (2020), with the bid-ask spread ignored in the first study and assumed exogenous in the second. Since the dealers possess information that is not available to end users, it is a legitimate question to ask whether the two sets of traders can be lumped together. We also know that the option market is partially segmented between puts and calls, as Constantinides and Lian (2020) have shown, implying that put-call parity does not hold.

In this paper we derive the SD bounds for the discretized version of a large class of bivariate SV models that includes all those that have appeared in the various NAE studies. In the principal result it is shown that in all cases the bounds tend at the limit to a *single* value of the option, thus demonstrating the ability of SD to derive results in the SV model without further assumptions, unlike the NAE methodology. The model is then transformed into a SVJ-SD model by including independent Poisson jumps, in which case at the continuous time limit the SD bounds tend to two different values and generate an interval of admissible option prices.

Although we do not carry out any empirical work in this paper, we use these SV-SD values and SVJ-SD bounds in order to evaluate their consistency with the extracted SV and SVJ parameters from published studies that use the NAE paradigm with SV or SVJ dynamics. Since in such studies the Q-parameters are extracted by fitting an SV or SVJ distribution to the observed bid-ask midpoint, this step is equivalent to examining whether that assumed equilibrium option value is consistent with the corresponding assessed P-dynamics. The motivation for this last step stems from Jouini and Kallal (1995), who examine market equilibrium in the presence of frictions for any type of financial assets and question (p. 181) the extraction of the frictionless risk neutral price of an asset from the observed bid-ask spread.[9] Preliminary results with short term options show that in a very large number of cross sections there is no overlap between the SD bounds and the observed bid-ask spread. This raises questions about the universally adopted assumptions for the intermediated market.

In the next section we formulate the SD model and present the option bounds for any discrete type of P-distribution, including the discretized SV and SVJ dynamics. Section 3 contains the principal result of this paper, the proof that the SV bounds converge to a single value, and

[9] See also the theoretical papers by Bizid *et al* (1999), Jouini (2003), and Bizid and Jouini (2005), whose approach has several common points with SD.

compares the SV-SD results using the formulation and P -parameter values of existing studies under the NAE paradigm with the extracted Q-distributions of these same studies. Section 4 presents SVJ-SD bounds as extensions of the constant volatility jump diffusion SD bounds and presents implications for future empirical research that would exploit the inconsistency of the frictionless SD results with the observed option market prices. Section 5 concludes**.**

II. The General Model and the SD Bounds[10]

Let S_t and K denote, respectively, the underlying index and option strike prices at any time $t = 0,1,...,T$ prior to option expiration in discrete time formulation. If Δt denotes the length of the time partition then in a single trading period $(t, t+\Delta t)$ the underlying asset with current price S_t has an ex-dividend rate of return $\dfrac{S_{t+\Delta t} - S_t}{S_t} \equiv z_{t+\Delta t}$. The riskless asset's return per period is equal to $R = e^{r\Delta t} = 1 + r\Delta t + o(\Delta t)$. Hereafter we denote by Index Trader or IT the generic risk averse investor holding the index and the riskless asset and by Option Trader or OT the same investor who also holds a zero net cost option or option portfolio.

Except for the trivial case where $z_{t+\Delta t}$ takes only two values the market for the index is incomplete in a discrete time context. The valuation of an option in such a market cannot yield a unique price. Our market equilibrium is derived under the following set of assumptions that are sufficient for our results:

There exists a utility-maximizing risk averse IT class of investors in the economy

These investors are marginal in the option market
The riskless rate is non-random

The IT investors optimize their portfolio holdings recursively over a horizon longer than the option maturity, at the end of which they maximize the expected utility of terminal wealth. The first order conditions of this maximization yield the pricing kernel $Y(z_{t+\Delta t})$, the state-contingent discount factor or normalized marginal rate of substitution of the trader evaluated at her optimal portfolio choice. Assuming no transaction costs and no taxes, the following relations characterize market equilibrium in any *single* trading period $(t, t+\Delta t)$,

$$E[Y(z_{t+\Delta t}) | S_t] = R^{-1}, \quad E[(1+z_{t+\Delta t})Y(z_{t+\Delta t}) | S_t] = 1 \ . \tag{2.1}$$

Because of the assumed risk aversion and portfolio composition of our traders it can be easily seen that the pricing kernel $Y(z_{t+\Delta t})$ would be monotone non-increasing) in the index return $z_{t+\Delta t}$ for every $t = 0,1,...,T$. These market equilibrium assumptions are quite general, insofar as they allow the existence of other investors with different portfolio holdings than the trader, although their application to option valuation implicitly assumes that these other investors will play a

[10] The material in this section summarizes results from Perrakis (2019, pp. 19-29).

limited role in the option market.

For univariate diffusion P-distributions the equilibrium conditions can be applied to the following ex-dividend return, the Euler discretization of a general diffusion process

$$z_{t+\Delta t} = \mu(S_t, t)\Delta t + \sigma(S_t, t)\varepsilon\sqrt{\Delta t}\,. \tag{2.2}$$

The random term ε has a distribution $F(\varepsilon)$ of bounded support, mean zero and variance one. Both the mean rate of return and its volatility are allowed to be functions of time and current asset price. The SD bounds, however, do not depend on the univariate diffusion assumption and can be expressed recursively in terms of a general distribution $P(z_{t+\Delta t}|S_t)$, which may be a convolution of the diffusion and an independent jump component, or may include other observable factors unrelated to the index return. In such a case, if we assume as is reasonable that $1+E[z_{t+\Delta t}|S_t] \geq R$ there exist upper and lower bounds $\overline{C}_t(S_t)$ and $\underline{C}_t(S_t)$ for European options given by the following recursive expressions if the option is assumed without loss of generality to be a call.

$$\begin{aligned}
&\overline{C}_T(S_T) = \underline{C}_T(S_T) = (S_T - K)^+ \\
&\overline{C}_t(S_t) = \frac{1}{R}E^{U_t}[\overline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t}))|S_t] \quad . \\
&\underline{C}_t(S_t) = \frac{1}{R}E^{L_t}[\underline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t}))|S_t]
\end{aligned} \tag{2.3}$$

In (2.3) E^{U_t} and E^{L_t} denote respectively expectations taken with respect to the following distributions

$$\begin{aligned}
&U_t(z_{t+\Delta t}) = \begin{cases} P(z_{t+\Delta t}|S_t) & \text{with probability} \quad \frac{R-1-z_{\min,t+\Delta t}}{E(z_{t+\Delta t})-z_{\min,t+\Delta t}} \\ 1_{z_{\min,t+\Delta t}} & \text{with probability} \quad \frac{E(z_{t+\Delta t})+1-R}{E(z_{t+\Delta t})-z_{\min,t+\Delta t}} \equiv \Theta \end{cases} \quad . \\
&L_t(z_{t+\Delta t}) = P(z_{t+\Delta t} \mid S_t, z_{t+\Delta t} \leq z_t^*),\ E(1+z_{t+\Delta t}|S_t, z_{t+\Delta t} \leq z_t^*) = R
\end{aligned} \tag{2.4}$$

The distributions $U_t(z_{t+\Delta t})$ and $L_t(z_{t+\Delta t})$ are both risk neutral and yield bounds on European option values through SD by successive integrations of the payoff as in (2.3), *provided* the option value is convex with respect to the price of the underlying, as it generally happens when the payoff is convex. SD can still derive bounds on derivatives when their payoff is not convex, but there are no closed form expressions for the bounds and the derivations are numerical.[11]

These SD recursive relations are sufficient to generate a single price for an option under SD for any state- and time-dependence of the mean and volatility of the discretized diffusion shown in (2.2), since the bounds can be shown to tend to the same value for $\Delta t \to 0$, given their dependence on a single random factor. In the case of SV, however, the discretization (2.2) is not sufficient, since the model is a bivariate diffusion. In fact, convergence holds here as well,

[11] See Perrakis and Boloorforoosh (2018) for an example of such derivatives.

producing a single option value, as shown in the following section.

III. The SD bounds Under Stochastic Volatility

We adopt a generalized formulation of the SV model, which to our knowledge includes the SV component of all the models that have appeared in the literature, most often combined with jumps. As in these models, neither the returns nor the volatility dynamics depend on S_t. Letting V denote the variance of the return, the P-dynamics of the index return are as follows

$$\begin{aligned} \frac{dS_t}{S_t} &= [r+\gamma(V)]dt + \sigma(V)dW_1, \\ dV &= \alpha(V)dt + \beta(V)dW_2,\ \ dW_1 dW_2 = \rho(V)dt \end{aligned} . \quad (3.1)$$

The discretization of these dynamics is now given by

$$\begin{aligned} z_{t+\Delta t} &= [r+\gamma(V_t)]\Delta t + \sigma(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t}, \\ V_{t+\Delta t} &= V_t + \alpha(V_t)\Delta t + \beta(V_t)[\rho(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t} + \sqrt{1-\rho^2(V_t)}\eta_{t+\Delta t}\sqrt{\Delta t}] \end{aligned} . \quad (3.2)$$

As before, the return shocks $\varepsilon_{t+\Delta t}$ have a distribution $F(\varepsilon)$ of bounded support, zero mean and unitary variance, while a distribution $G(\eta)$ with the same properties also holds for the independent idiosyncratic variance shocks $\eta_{t+\Delta t}$. The correlation coefficient $\rho(V_t)$, the so-called leverage effect, has consistently been shown to be negative, as it will be assumed from now on. Hence, the covariance matrix at *t* of the vector $(z_{t+\Delta t}, V_{t+\Delta t})$ is equal to

$$\begin{bmatrix} \sigma^2(V_t) & \rho(V_t)\sigma(V_t)\beta(V_t) \\ \rho(V_t)\sigma(V_t)\beta(V_t) & \beta^2(V_t) \end{bmatrix} \Delta t \equiv \mathrm{M}_t^P(V)\Delta t . \quad (3.3)$$

The next step is the derivation of the equilibrium relation (2.1) consistent with the discretized index dynamics (3.2). Here the kernel has the form $Y(z_{t+\Delta t}, V_{t+\Delta t})$, a function of both return and volatility. To apply SD we need conditions under which the function $E_t[Y(z_{t+\Delta t}, V_{t+\Delta t}) | S_{t+\Delta t}] \equiv \widehat{Y}_t(z_{t+\Delta t})$ is monotone non-increasing. As noted in the introduction, this issue is controversial and has been debated for more than 20 years, in studies that use SV or GARCH *P*-dynamics but reach diametrically opposite conclusions even though they used virtually identical time series data. There should not have been any controversy, since the theoretical arguments for monotonicity are compelling, and the fact that the non-monotonicity claims are based solely on empirical work should have raised a debate about that work, rather than non-monotonicity.[12] As Wiggins (1987, pp. 356-358) has argued by citing Merton (1971), explicit forms of $Y(z_{t+\Delta t}, V_{t+\Delta t})$ exist only for time-additive constant relative risk aversion

[12] See Perrakis (forthcoming, 2022) for an extensive presentation of the debate. Christoffersen, Heston and Jacobs (2013, pp. 1966-1967) formulate a non-monotone kernel in the Heston (1993) SV and in the GARCH Heston and Nandi (2001) models without a formal theoretical justification, but also state that the risk neutral SV dynamics have identical functional forms for both monotone and non-monotone versions of the SV kernel.

(CRRA) utilities, in which the value function of the representative investor consists of the product of two terms, one involving the volatility and the other the index return. The term involving the volatility is increasing (decreasing) if the relative risk aversion (RRA) coefficient is greater (less) than 1. Since the empirically most relevant case is an RRA greater than 1, the term that contains the volatility is increasing in $V_{t+\Delta t}$, implying by the leverage effect that the function $\widehat{Y}_t(z_{t+\Delta t})$ is non-increasing. Further, Beare (2011) shows on the basis of a result originally proven by Dybvig (1988), that when the kernel is not counter-monotone any investor can replicate at a lower cost the index with option portfolios. If the investor then replaces in her holdings the index by the option portfolio there is first degree stochastic dominance and a violation of the key no arbitrage assumption. This replication is particularly easy for the Christoffersen, Heston and Jacobs (2013) U-shaped kernel for GARCH dynamics, where it can be shown that with such a kernel a portfolio of a long call, a short put with the same strike price and a long position in the riskless asset overvalues the index and violates no arbitrage.

At $T-\Delta t$ the distribution $P(z_T \,|S_{T-\Delta t})$ can be easily extracted from the first line of (3.2) given $F(\varepsilon)$. Hence, the option boundary distributions (2.4) apply directly and the option bounds are $\overline{C}_{T-\Delta t}(S_{T-\Delta t},V_{T-\Delta t})=E^U_{T-\Delta t}[(S_T-K)^+]$ and $\underline{C}_{T-\Delta t}(S_{T-\Delta t},V_{T-\Delta t})=E^L_{T-\Delta t}[(S_T-K)^+]$. However, while the option payoff $(S_T-K)^+$ is obviously convex and does not involve the idiosyncratic volatility shock η_T, none of these properties holds for the option value at any $t<T-\Delta t$. We apply induction, assuming that $\overline{C}_{t+\Delta t}(S_{t+\Delta t},V_{t+\Delta t})$ and $\underline{C}_{t+\Delta t}(S_{t+\Delta t},V_{t+\Delta t})$ are known and $\widehat{Y}_t(z_{t+\Delta t})$ is non-increasing, and seeking to define bounds at t such that $\underline{C}_t(S_t,V_t)\le C_t(S_t,V_t)\le \overline{C}_t(S_t,V_t)$. These bounds must satisfy the following equilibrium problem.

$$\begin{aligned}
&Max_{\widehat{Y}}\{E_t[\widehat{Y}_t(z_{t+\Delta t})\overline{C}_{t+\Delta t}(S_{t+\Delta t},V_{t+\Delta t})]\},\;\; Min_{\widehat{Y}}\{E_t[\widehat{Y}_t(z_{t+\Delta t})\underline{C}_{t+\Delta t}(S_{t+\Delta t},V_{t+\Delta t})]\}\\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ,\\
&E_t[\widehat{Y}_t(z_{t+\Delta t})\,|S_t]=R^{-1},\;\; E_t[(1+z_{t+\Delta t})\widehat{Y}_t(z_{t+\Delta t})\,|S_t]=1
\end{aligned} \tag{3.4}$$

where the kernel $\widehat{Y}_t(z_{t+\Delta t})$ is non-increasing.

In what follows we *assume* initially that the Q-distribution is such that the option value $C_t(S_t,V_t)$ is convex in S_t. Conditions for such convexity were derived in Bergman, Grundy and Wiener (1996), and the P-dynamics in (3.1) satisfy them. Our assumption will be justified if our derived Q-dynamics also satisfy them. In the program (3.4) the dependence of $C_{t+\Delta t}(S_{t+\Delta t}(\varepsilon),V_{t+\Delta t}(\varepsilon,\eta))$ on the random shocks ε and η obviously carries over in its bounds. We also assume without loss of generality that the distribution of the shock ε is discrete, $(p_i,\varepsilon_i),\ i=1,...,n$ in ascending order. Then we can prove the following result.

Lemma 1: Under the induction hypothesis, at every $t < T - \Delta t$ we have $\underline{C}_t(S_t, V_t) \le C_t(S_t, V_t) \le \overline{C}_t(S_t, V_t)$, where $\underline{C}_t(S_t, V_t)$ and $\overline{C}_t(S_t, V_t)$ are convex and do not depend on the volatility shock η and are given by

$$\begin{aligned} \underline{C}_t(S_t, V_t) &= \int_{\underline{\eta}}^{\overline{\eta}} E_t^L[\underline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon, \eta))dG(\eta) \\ \overline{C}_t(S_t, V_t) &= \int_{\underline{\eta}}^{\overline{\eta}} E_t^U[\overline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon, \eta))dG(\eta) \end{aligned} \quad . \tag{3.5}$$

In (3.5) the expectations with respect to the shock ε within the integrals are given by the following expressions for each value of the idiosyncratic volatility shock η, and $[\underline{\eta}, \overline{\eta}]$ is the support of that shock.

$$\begin{aligned} & E_t^L[\underline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon, \eta))] = \sum_{i=1}^{i=h+1} l_i \underline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon_i), V_{t+\Delta t}(\varepsilon_i, \eta)) \\ & l_i = \vartheta \frac{p_i}{\sum_{j=1}^{j=h+1} p_j} + (1-\vartheta) \frac{p_i}{\sum_{j=1}^{j=h} p_j}, \ i = 1, ..., h, \\ & l_{h+1} = \frac{p_i}{\sum_{j=1}^{j=h} p_j}, \ l_i = 0, \ i = h+2, ..., n, \ \vartheta = \frac{R - 1 - \hat{z}_h}{\hat{z}_{h+1} - \hat{z}_h}, \\ & 1 + \hat{z}_h^L \le R \le 1 + \hat{z}_{h+1}^L, \ \hat{z}_i^L = E_t^L[z(\varepsilon_j) \,|\, j \le i], \ \vartheta \hat{z}_{h+1}^L + (1-\vartheta)\hat{z}_h^L = R - 1 \end{aligned} \quad , \tag{3.6}$$

and similarly for the upper bound

$$\begin{aligned} & E_t^U[\overline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon, \eta))] = \sum_{i=1}^{i=n} u_i \overline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon_i), V_{t+\Delta t}(\varepsilon_i, \eta)) \\ & u_1 = \theta p_1 + (1-\theta), \quad u_i = (1-\theta) p_i, \quad i = 2, ..., n, \quad \theta = \frac{1 + E_t[z_{t+\Delta t}] - R}{E_t[z_{t+\Delta t}] - z_1} \text{ as in (2.4),} \\ & \hat{z}_i^U = E_t^U[z_j \,|\, j \le i], \quad \theta \hat{z}_n^U + (1-\theta) z_1 = \theta E_t[z_{t+\Delta t}] + (1-\theta) z_1 = R - 1 \end{aligned} \tag{3.7}$$

Proof: See Appendix A.

The following expressions also follow directly from (3.6)-(3.7) and (A.1)-(A.2).

$$\begin{aligned} \underline{C}_t(S_t, V_t) &= R^{-1} \int_{\underline{\eta}}^{\overline{\eta}} [\vartheta \hat{\underline{C}}_{h+1, t+\Delta t}(\eta) + (1-\vartheta) \hat{\underline{C}}_{h, t+\Delta t}(\eta)] dG(\eta), \\ \overline{C}_t(S_t, V_t) &= R^{-1} \int_{\underline{\eta}}^{\overline{\eta}} [\theta \hat{\overline{C}}_{n, t+\Delta t}(\eta) + (1-\theta) \hat{\overline{C}}_{1, t+\Delta t}(\eta)] dG(\eta) \end{aligned} \quad . \tag{3.8}$$

Observe that the two volatility shocks ε and η have joint distribution $F(\varepsilon)G(\eta)$, which implies that the order of the estimations in (3.5)-(3.7) can be reversed, by integrating out the idiosyncratic volatility shock η. This may present some advantages in implementing the algorithm, by avoiding corner solutions in intermediate steps.

We have thus shown that in discretized time the option value is bound by two convex functions that do not depend on η, derived recursively from a monotone pricing kernel. The last and most important result of this section has the following form.[13]

Proposition 1: For any $t < T$ and for $\Delta t \to 0$ the option bounds for the SV model (3.1), defined in (3.5) and (3.8), converge to the unique option price given by the expectation of the payoff with the following Q-dynamics.

$$\begin{aligned} &\frac{dS_t}{S_t} = rdt + \sigma(V_t)dW_1^Q, \\ &dV = \left(\alpha(V_t) - \frac{\rho(V_t)\beta(V_t)\gamma(V_t)}{\sigma(V_t)}\right)dt + \beta(V_t)[\rho(V_t)W_1^Q + \sqrt{1-\rho^2(V_t)}W_2^Q] \end{aligned} \quad . \tag{3.9}$$

Observe[14] that $dW_1^Q = dW_1 + \dfrac{\gamma(V_t)}{\sigma(V_t)}dt, \;\; dW_2^Q = dW_2$.

Proof: For any $t < T$ define $\Delta V = V_{t+\Delta t} - V_t$ and let $R - 1 = r$. Consider the discretization (3.2) of the P-distribution, for which we have $E_t^P\begin{bmatrix} z_{t+\Delta t} \\ \Delta V \end{bmatrix} = \begin{bmatrix} r+\gamma(V) \\ \alpha(V) \end{bmatrix}\Delta t$. Then for the two Q-bounds, for which we have $E_t^Q[z_{t+\Delta t}] = r\Delta t$, we must also have $\sigma(V)E_t^Q[\varepsilon_{t+\Delta t}] = -\gamma(V)\sqrt{\Delta t}$. Taking the Q-expectation of $\Delta V = \alpha(V_t)dt + \beta(V_t)[\rho(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t} + \sqrt{1-\rho^2(V_t)}\eta_{t+\Delta t}\sqrt{\Delta t}]$ and replacing $E_t^Q[\varepsilon_{t+\Delta t}]$ by its equal, we find that $E_t^Q\begin{bmatrix} z_{t+\Delta t} \\ \Delta V \end{bmatrix} = \begin{bmatrix} r \\ \alpha(V) - \dfrac{\rho(V)\beta(V)\gamma(V)}{\sigma(V)} \end{bmatrix}\Delta t$.

For the covariance matrix $\mathrm{M}_t^Q(V)$, the equivalent of (3.3), the proof of convergence differs between the upper and the lower bound. The full proof is available in Oancea and Perrakis (2014, appendix) and will only be summarized here. For the upper bound it can be shown that the parameter θ defined in (3.7) tends to $-\dfrac{\gamma(V)}{\sigma(V)\varepsilon_{\min}}\sqrt{\Delta t}$. This implies, since $Var^P[\varepsilon] = 1$, that

[13] An alternative proof, available from the authors on request, derives the Q-dynamics (3.9) by a linear programming (LP) approach without using Lemma 1.

[14] In other words, $E_t^P[dW_1^Q] = \dfrac{\gamma(V_t)}{\sigma(V_t)}dt, \;\; E_t^Q[dW_1^Q] = 0$.

$\lim_{\Delta t\to 0} Var_t^U[z_{t+\Delta t}] = \sigma^2(V)\Delta t\left[\left(1+\frac{\gamma(V)}{\sigma(V)\varepsilon_{\min}}\sqrt{\Delta t}\right)-\frac{\gamma(V)}{\sigma(V)\varepsilon_{\min}}\sqrt{\Delta t}\varepsilon_{\min}^2\right] = \sigma^2(V)\Delta t + o(\Delta t)$. For the lower bound the proof is simpler when the distribution $F(\varepsilon)$ is continuous, as assumed in (2.4). In such a case we obviously have $E_t^L[\varepsilon_{t+\Delta t}] = -\frac{\gamma(V)}{\sigma(V)}\sqrt{\Delta t} = \frac{1}{F(\bar{\varepsilon})}\int_{\varepsilon_{\min}}^{\bar{\varepsilon}} \varepsilon dF(\varepsilon)$, where $\bar{\varepsilon}(\Delta t) < \varepsilon_{\max}$. It can be shown that $\lim_{\Delta t\to 0}[1-F(\bar{\varepsilon})]=0$, from which it turns out that $\lim_{\Delta t\to 0} Var_t^L(\varepsilon) = 1$. Hence, for $\Delta t \to 0$ and for both covariance matrices we have $\mathrm{M}_t^Q(V) = \mathrm{M}_t^P(V)$, and by the Lindeberg condition, observing that dW_1^Q is a martingale under the Q-distribution, we get (3.9). Since these dynamics justify the Bergman, Grundy and Wiener (1996) conditions for convexity, the initial convexity assumption is justified, QED.

For comparison purposes, we rewrite (3.1) by decomposing the volatility shocks into two uncorrelated Wiener processes

$$\begin{aligned} &\frac{dS_t}{S_t} = [r+\gamma(V_t)]\mu dt + \sigma(V_t)dW_1, \\ &dV_t = \alpha(V_t)dt + \beta(V_t)[\rho(V_t)dW_1 + \sqrt{1-\rho^2(V_t)}dW_2],\ \ E[dW_1 dW_2]=0 \end{aligned} \qquad . \tag{3.1'}$$

This allows us to formulate the following important result, whose proof is obvious from (3.1)' and (3.9) if we define (dV^P, dV^Q) respectively from these dynamics, in which case the realized variances over an interval $[t,T]$ are given by $\int_t^T E_t(dV_\tau^I)d\tau,\ I = P,Q$.

<u>Corollary</u>: The volatility spreads over the maturity of an option are given by

$$\frac{\int_t^T E_t(dV_\tau^Q)d\tau - \int_t^T E_t(dV_\tau^P)d\tau}{\int_t^T E_t(dV_\tau^P)d\tau} = \frac{\int_t^T (\hat{\alpha}(V_\tau)-\alpha(V_\tau))d\tau}{\int_t^T \alpha(V_\tau))d\tau} = \frac{\int_t^T -\frac{\rho(V_\tau)\beta(V_\tau)\gamma(V_\tau)}{\sigma(V_\tau)}d\tau}{\int_t^T \alpha(V_\tau))d\tau}. \tag{3.10}$$

These volatility spreads are obviously directly related to the traded variance swaps, except for the fact that they do not require option market data. They can also be compared to the results of Section 4 of Bakshi and Madan (2004), who study volatility spreads in a general setup that does not assume a particular form of *P*-dynamics for the index. Instead, they extract their results from an equilibrium formulation with a CRRA investor and a quadratic expansion in Taylor series of the pricing kernel. By contrast, in our results the derivation of the *Q*-dynamics are uniquely defined from the *P*-dynamics and do not require further information.

The last result of this section derives the pricing kernel from the *P*- and *Q*-dynamics of Proposition 1. This result is not necessary, since Proposition 1 derived independently the risk neutral dynamics, but is presented here for comparison purposes, since the kernel is fundamental for many studies that use the NAE methodology and derive the Q-distribution from the option market by fitting a kernel that reconciles the physical and risk neutral dynamics. It is expressed by the following result, which uses the derivations of Proposition 1 to extract the kernel.

Proposition 2: The stochastic volatility pricing kernel for the P- and Q- dynamics shown in (3.1') and (3.9) respectively is given by the following relation

$$\begin{aligned}&\hat{Y}(z_{t+\Delta t}^{P}) = \exp[\frac{1}{2}(\frac{\gamma(V_t)}{\hat{\sigma}})(2\mu^P - \frac{\gamma(V_t)}{\hat{\sigma}} - 2\hat{z}^P)] \Rightarrow \\ &\ln \hat{Y} = \frac{1}{2}(\frac{\gamma(V_t)}{\hat{\sigma}})(2\mu^P - \frac{\gamma(V_t)}{\hat{\sigma}} - 2\hat{z}^P)\end{aligned}, \quad (3.11)$$

Where

$$\begin{aligned}&\hat{z}^P = z\frac{\beta(V_t)\rho(V_t)-\sigma(V_t)}{\beta(V_t)\rho(V_t)\sigma(V_t)} \equiv \frac{z}{\hat{\sigma}}, \quad \hat{z}^Q = \hat{z}^P + \frac{\gamma(V_t)}{\hat{\sigma}} \\ &\mu^P = \frac{r+\gamma(V_t)}{\sigma(V_t)} - \frac{\alpha(V_t)}{\beta(V_t)\rho(V_t)} - 0.5[\sigma(V_t) - \beta(V_t)\rho(V_t)] = \\ &\frac{[r+\gamma(V_t)]\beta(V_t)\rho(V_t) - \alpha(V_t)\sigma(V_t)}{\hat{\sigma}[\beta(V_t)\rho(V_t)-\sigma(V_t)]} - \frac{0.5[\sigma(V_t)-\beta(V_t)\rho(V_t)]\hat{\sigma}}{\hat{\sigma}}. \\ &\mu^Q = \frac{r}{\sigma(V_t)} - \frac{\hat{\alpha}(V_t)}{\beta(V_t)\rho(V_t)} - 0.5[\sigma(V_t) - \beta(V_t)\rho(V_t)] = \\ &\frac{r\beta(V_t)\rho(V_t) - \hat{\alpha}(V_t)\sigma(V_t)}{\hat{\sigma}[\beta(V_t)\rho(V_t)-\sigma(V_t)]} - \frac{0.5[\sigma(V_t)-\beta(V_t)\rho(V_t)]\hat{\sigma}}{\hat{\sigma}}\end{aligned} \quad (3.12)$$

Proof: See Appendix B.

From (3.11) and (3.12) we also get the following, whose proof is obvious.

Corollary: The continuous time kernel dynamics are, from (3.11)

$$d\ln\hat{Y} = \frac{d\hat{Y}}{\hat{Y}} = (-\frac{\gamma(V_t)}{\hat{\sigma}})dt + \frac{\sigma(V_t)}{\hat{\sigma}}dW_1. \quad (3.13)$$

From the proof of Proposition 2 in the appendix it follows that the dynamics in (3.13) define a $N(-\frac{\gamma(V_t)}{\hat{\sigma}}dt, \sqrt{dt})$ random variable. Note that the random factor $dW_2^Q = dW_2$ does not enter into the kernel, since the latter is derived by the ratio of the Q- to P- probabilities and the volatility shocks that do not affect the return cancel out.

Equation (3.11) is consistent with the exponential kernel in Theorem 1 of Bakshi and Madan (2004, p. 1949), which was assumed to correspond to the power utility of a representative investor. In our notation their kernel is given by $\psi_{0t}\exp(-\psi_{1t}z_{t+\Delta t}^{P})$, which is also the form adopted on an ad hoc basis by Rosenberg and Engle (2002, p. 347) for GARCH dynamics. The coefficient ψ_{1t}, which corresponds to the RRA of the representative investor, is equal to $\hat{\sigma}^{-1}$ and is time-and state-varying.

Except for the change in notation, the P-dynamics in (3.1)-(3.2) are generalized versions of the ones used in Fournier and Jacobs (2020, p. 1122). The simplicity of the risk neutralization of Proposition 1 should be contrasted with the approach used in this latter study, which derives the Q-distribution in an equilibrium argument that includes a monopolistic market maker in the option market and a third state variable consisting of the inventory to wealth ratio of that agent. This state variable enters into the risk neutral dynamics shown in equations (35)-(37) of that study, which should be compared to the risk neutralization under SD of the model shown in equation (3.14b) below, whose P-dynamics are identical to those of the Fournier-Jacobs equations (1) and (2). See, in particular, their equation (36), in which $E^Q[dV_t]$ includes the inventory to wealth ratio, which must be estimated from the option market. Unfortunately, numerical comparisons between their results and ours cannot be done with the material presented in their study.

Such a comparison, though, is feasible for the results of Jones (2003), who studied the S&P 100 index and considered a variety of models starting with the modified Heston (1993) SV, termed the SQRT model,[15] and generalized or modified it along several dimensions. The impact of our endogenous determination of volatility risk can best be judged by comparing its resulting Q-dynamics with the ad hoc estimation of these same dynamics in the Jones study, by fitting to the observed option market data. Thus, omitting the time subscript, in the SQRT model we have

$$\begin{aligned}&\gamma(V)=\mu-r,\ \sigma(V)=\sqrt{V},\ \alpha(V)=\zeta(V_0-V),\\&\beta(V)[\rho(V)dW_1+\sqrt{1-\rho^2(V))}dW_2]=\sigma_v\sqrt{V}[\rho dW_1+\sqrt{1-\rho^2}dW_2]\end{aligned}. \quad (3.14a)$$

Similarly, in Jones' first extension, termed the CEV these dynamics become

$$\begin{aligned}&\gamma(V)=\mu-r,\ \sigma(V)=\sqrt{V},\ \alpha(V)=\varphi+\beta V\\&\beta(V)=\sigma_v V^{\varsigma},\ \sigma_v=\sqrt{\sigma_1^2+\sigma_2^2},\ \rho=\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\end{aligned}. \quad (3.14b)$$

In Jones' second extension, the 2GAM model, we have

$$\begin{aligned}&\gamma(V)=\mu-r,\ \sigma(V)=\sqrt{V},\ \alpha(V)=a+bV\\&\beta(V)[\rho(V)dW_1+\sqrt{1-\rho^2(V)}dW_2]=\sigma_1V^{\varsigma_1}dW_1+\sigma_2V^{\varsigma_2}dW_2\Rightarrow\\&\beta(V)=\sqrt{(\sigma_1V^{\varsigma_1})^2+(\sigma_2V^{\varsigma_2})^2},\quad \rho(V)=\frac{\sigma_1V^{\varsigma_1}}{\sqrt{(\sigma_1V^{\varsigma_1})^2+(\sigma_2V^{\varsigma_2})^2}}\end{aligned}. \quad (3.14c)$$

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[15] In Heston (1993) only the Q-dynamics were presented, while the kernel was only derived in 2013, by Christoffersen *et al* (2013), for the case where $\gamma(V)=\xi V$. Jones' SQRT model has $\gamma(V)=\mu-r$.

Hence, the variance risk premium, reflecting the price of volatility risk, becomes in these three cases equal to $-\rho(\mu-r)\sigma_v$ for SQRT, $-\rho\xi\sigma_v V$ for the Heston model as in Christoffersen *et al* (2013), to $\dfrac{-\rho(\mu-r)\sigma_v V^{\varsigma}}{\sqrt{V}}$ for the CEV, and to $\dfrac{\sigma_1 V^{\varsigma_1}(\mu-r)}{\sqrt{(\sigma_1 V^{\varsigma_1})^2+(\sigma_2 V^{\varsigma_2})^2}}$ for 2GAM. This endogenous variance risk premium is appropriately redefined if the index risk premium $\mu-r$ in the Jones models is set equal to $\xi\sqrt{V}$ or ξV. For the Q-dynamics Jones replaces $a+bV$ by $a+b^*V$ and leaves the other parameters unchanged. He also defines (p. 187) the implied price of variance risk as $\dfrac{(b-b^*)V}{\sqrt{(\sigma_1 V^{\varsigma_1})^2+(\sigma_2 V^{\varsigma_2})^2}}=\dfrac{(b-b^*)V}{\beta(V)}$. The major economic difference from our endogenous SV variance risk premium is that the leverage effect and the equity risk premium are not taken into account in the equilibrium determination of the price of volatility risk, shown in Appendix C of the Jones study.[16]

Tables 1, 2 and 3 below show the differences in option valuation between the endogenous volatility risk of the SD approach and the one derived by fitting the Q-dynamics of the SV model to the noisy S&P 100 index option market data under the SQRT, CEV and 2GAM models, respectively. In all three tables the P-parameters are the ones shown in Columns 2 and 4 of Table 1 of Jones (2003, p. 197), corresponding to the 1988-2000 time series. We use the format of Table 5 of Jones (2003, p. 207), with the tables showing the option values corresponding to a \$100 price of the index, with strike prices equal to the ones shown on the top line; only OTM values are shown, puts on the right and calls on the left. The maturities T in days are the ones chosen by Jones, as are the starting volatilities V_0. In all tables the top panel termed NAE shows the values obtained by the closed form expressions of the Heston (1993) model using Jones' b^* value. The bottom SD panel shows the option values according to the Q-dynamics in (3.14abc), corresponding to the endogenous price of volatility risk of the corresponding model with a constant risk premium. The Table 1 SD results were very similar when the risk premium was set proportional to the variance as in Christoffersen *et al* (2013). The Monte Carlo method was used for Tables 2 and 3. The standard errors were in all cases lower than the 2% mentioned by Jones (p. 207) for all but the very low priced deep OTM options.

[Table 1 about here]

The comparative results for the SQRT model are striking. Our fitted option values based on the empirically extracted b^* value reproduce accurately Jones' results and are in *all* cases above, and often far above, the theoretical frictionless SD prices. The differences between the NAE and SD prices are minor in the ATM and shortest maturity options, but they escalate dramatically as the options become deeper OTM for both calls and puts. More to the point, they increase sharply at

[16] By contrast, the leverage effect appears in Fournier and Jacobs (2020, p. 1129). In fact, the first term in the variance risk premium of expression (29) in that paper is equal to the SD risk premium, without any need for the second term that involves the inventory to wealth ratio.

higher maturities, with the ATM 22-day option price under the 0.001 volatility higher by close to 17% and the 66-day price higher by a stunning 50% under NAE than under SD.

[Table 2 about here]

[Table 3 about here]

Tables 2 and 3 show the same relationships in the comparative results between the fitted NAE and the theoretically estimated SD values for the CEV and 2GAM models as with SQRT. To begin with, our NAE values reproduce very accurately the corresponding values of Jones' Table 5, with a few exceptions for the 66-day maturity where, however, the differences are not significant in terms of the standard errors. As with Table 1, in all cases the SD values are significantly lower than the corresponding NAE for all but the ATM options in the 5-day maturity. As for the comparison of the results for the three SV models, the comparative results in the SV panel of the three tables show major differences between the models, with the sign of the differences dependent on moneyness and maturity.

A full explanation of these results requires significant empirical work with more recent data and lies beyond the scope of this paper. Although the Jones article does not specify how the option values were selected from the data, it is most probable that they were equal to the observed option bid-ask spread midpoint, as in most empirical option research. Further, the strong maturity effect that we observe in all tables most probably reflects the greater importance of volatility in longer maturity options. The large differences NAE-SD most probably imply that the SD prices lie below the observed bid prices in the option market in all cases. Although we do not have data for the S&P 100 option market bid-ask spreads, we do have some indications from Perrakis (2022, Figure 1) that for S&P 500 options the bid-ask spread is proportionately much higher for OTM and shorter maturity options. Still, the observed magnitude of the NAE-SD difference at ATM which was noted for SQRT is even higher under CEV and 2GAM than the 16.9% of the SQRT model at the 22-day maturity. At the same time the median observed bid-ask spread for the S&P 500 28-day maturity options was around 5% and never exceeded 10% over the 1990-2000 time period.

In fact the unreasonably high values implied by the NAE models were also noted by Jones (2003, pp. 207-208), who states that "the large positive drift in the CEV variance process implies very expensive options of all degrees of moneyness". He also conjectured that these high option prices may have been due to misspecification of the volatility risk. A more likely explanation is the inconsistency of the option market data with the P-dynamics extracted from observed index return data. As noted in the introduction, the option market is an intermediated market that has never been modelled in detail but is assumed in the frictionless NAE models of Garleanu, Petersen and Poteshman (2009) and Fournier and Jacobs (2020) to consist of passive liquidity providers operating, respectively, in a perfectly competitive market and in a monopolistic market with exogenous bid-ask spreads. A major advantage of the SD endogenous determination of the variance risk premium in the SD approach is the fact that it allows us to test these intermediate market assumptions, as discussed in the last section of this paper.

IV. The SD bounds Under Stochastic Volatility and Jumps

The SD bounds for index options under constant volatility jump diffusion were analyzed in detail in Perrakis (2019, pp. 45-61), in which it was shown that the market is incomplete even in that simple case, with the SD bounds converging to two different values. We expect, therefore, that a similar result will also prevail under SV. We use again the very general SV formulation in (3.1) and its discretized version (3.2) and augment them with independent Poisson jumps in the index return with intensity equal to λ and random log-amplitude J, whose distribution is given by $J = \ln(j) \sim D(\mu_j - \frac{1}{2}\sigma_j^2, \sigma_j)$, with $\mu_j = E[\ln j]$, $\kappa = e^{\mu_j} - 1$. Although this distribution is assumed normal in most empirical applications, we shall not adopt this assumption, which is not realistic for all but the very lengthy option maturities.[17] Instead, we shall assume that there exists a minimum value $j_{\min} > 0$ of the amplitude, which translates into a minimum value $J_{\min} > -\infty$ of the rate of return, at which the lognormal is truncated. In such a case the P-dynamics of the index in (3.1) become

$$\begin{aligned} \frac{dS_t}{S_t} &= [r + \gamma(V) - \lambda\kappa]dt + \sigma(V)dW_1 + (j-1)d\mathrm{N}, \\ dV &= \alpha(V)dt + \beta(V)dW_2, \; dW_1 dW_2 = \rho(V)dt \end{aligned} , \tag{4.1}$$

Where N is a Poisson counting process with intensity λ. We shall also assume, as is reasonable, that $j_{\max} > 1$, or $J_{\max} > 0$. The discretized version of (4.1) now becomes

$$\begin{aligned} z_{t+\Delta t} &= \begin{pmatrix} [r + \gamma(V_t) - \lambda\kappa]\Delta t + \sigma(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t} \text{ with probability 1-}\lambda\Delta\text{t} \\ [r + \gamma(V_t) - \lambda\kappa]\Delta t + \sigma(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t} + (j-1) \text{ with probability } \lambda\Delta t \end{pmatrix}, \\ V_{t+\Delta t} &= V_t + \alpha(V_t)dt + \beta(V_t)[\rho(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t} + \sqrt{1-\rho^2(V_t)}\eta_{t+\Delta t}\sqrt{\Delta t}] \end{aligned} . \tag{4.2}$$

This SVJ model is the simplest and most parsimonious in terms of its jump component that has appeared in the literature. The derivation of its Q-dynamics can be extended with very little reformulation to more complex cases, when the intensity becomes a linear function of the variance or volatility. More complex, but still tractable with the approach adopted in this paper, is the inclusion of jumps into the volatility equation, for which the SD derivation of the bounds will change. Since parsimoniousness of a model is a desirable property, we shall not proceed beyond the SVJ in (4.1). The proofs will only be sketched, since SVJ is a straightforward combination of SV as presented in the previous section and the constant volatility jump diffusion SD bounds.

To derive the Q-distribution(s) under SVJ we start from the model-free relations (2.3)-(2.4). The proof relies heavily on the constant volatility jump diffusion bounds under SD and is done

[17] SD bounds also exist when the jump amplitude is allowed to be full lognormal, but they are no longer both risk neutral payoff expectations.

separately for the upper and lower bounds. At time $T-\Delta t$ the martingale transformation of the upper bound $U_{T-\Delta t}(z_T)$ clearly does not involve the diffusion component, which stays the same, since there exists an *h*, such that for any $\Delta t \le h$, the minimum outcome of the jump component is less than the minimum outcome of the diffusion component, or $(j_{\min}-1) < ([r+\gamma(V_{T-\Delta t})-\lambda\kappa]\Delta t + \sigma(V_{T-\Delta t})\varepsilon_{\min}\sqrt{\Delta t})$.. Hence, as shown in Perrakis (2019, p. 49) and omitting the terms of the form $o(\Delta t)$ the Θ-probability in (2.4) becomes equal to $-\frac{\gamma(V_{T-\Delta t})}{(j_{\min}-1)}\Delta t \equiv \lambda_{UT}\Delta t$, and the jump component is a mixture $j_T^U = \begin{cases} j & \text{with probability } \frac{\lambda}{\lambda+\lambda_{UT}} \\ j_{\min} & \text{with probability } \frac{\lambda_{UT}}{\lambda+\lambda_{UT}} \end{cases}$, with mean amplitude

$E\left[j_T^U - 1\right] = \kappa_T^U = (\frac{\lambda}{\lambda+\lambda_{UT}})\kappa + (\frac{\lambda_{UT}}{\lambda+\lambda_{UT}})\left(j_{\min}-1\right)$. The return (4.2) now becomes

$$z_T^U = \begin{pmatrix} [r+\gamma(V_{T-\Delta t})-(\lambda+\lambda_{UT})\kappa_{t+\Delta t}^U]\Delta t + \sigma(V_{T-\Delta t})\varepsilon_T\sqrt{\Delta t} \text{ with prob. 1-}(\lambda+\lambda_{UT})\Delta \text{t} \\ [r+\gamma(V_t)-(\lambda+\lambda_{UT})\kappa_{t+\Delta t}^U]\Delta t + \sigma(V_{T-\Delta t})\varepsilon_T\sqrt{\Delta t} + (j_T^U - 1) \text{ with prob. } (\lambda+\lambda_{UT})\Delta t \end{pmatrix}, \quad (4.3)$$
$$V_T = V_{T-\Delta t} + \alpha(V_{T-\Delta t})dt + \beta(V_{T-\Delta t})[\rho(V_{T-\Delta t})\varepsilon_T\sqrt{\Delta t} + \sqrt{1-\rho^2(V_{T-\Delta t})}\eta_T\sqrt{\Delta t}]$$

Hence, $\overline{C}_{T-\Delta t}(S_{T-\Delta t}, V_{T-\Delta t}) = E_{T-\Delta t}^U[(S_T-K)^+] \ge C(S_{T-\Delta t}, V_{T-\Delta t})$. A similar relation also holds for the lower bound $\underline{C}_{T-\Delta t}(S_{T-\Delta t}, V_{T-\Delta t}) = E_{T-\Delta t}^L[(S_T-K)^+] \le C(S_{T-\Delta t}, V_{T-\Delta t})$, for which risk neutralization is achieved by truncating the jump amplitude at the right tail, as in Perrakis (2019, pp. 51-52), yielding $\begin{aligned} \gamma(V_{T-\Delta t}) &= \lambda\kappa - \lambda\kappa_T^L \\ \lambda\kappa_T^L &= E(j-1 \mid j \le \overline{j}_T) \end{aligned}$.

The SVJ bounds can now be derived recursively for any $t<T$ as in (2.3), given the fact that the P-dynamics of the return are a convolution or mixture of two independent processes, the discretized bivariate diffusion of the previous section and the jump process as in (4.1)-(4.2). The key issue is the derivation of the martingale probabilities (2.4) for the mixed process. Applying induction given $\underline{C}_{t+\Delta t}^L(S_{t+\Delta t}, V_{t+\Delta t}) \le C_{t+\Delta t}(S_{t+\Delta t}, V_{t+\Delta t}) \le \overline{C}_{t+\Delta t}^U(S_{t+\Delta t}, V_{t+\Delta t})$, the probabilities (2.4) in the absence of a jump at $t+\Delta t$ are found from the equilibrium relations (3.4), which still hold in modified version for the bivariate diffusion returns $z_{D,t+\Delta t}$

$$\begin{aligned} &Max_{\hat{Y}}\{E_t[\widehat{Y}_t(z_{D,t+\Delta t})\overline{C}_{t+\Delta t}^U(S_t(1+z_{D,t+\Delta t}), V_{t+\Delta t})]\}, \;\; Min_{\hat{Y}}\{E_t[\widehat{Y}_t(z_{Dt+\Delta t})\underline{C}_{t+\Delta t}^L(S_t(1+z_{D,t+\Delta t}), V_{t+\Delta t})]\} \\ &\text{subject to} \\ &E_t[\widehat{Y}_t(z_{D,t+\Delta t}) \big| S_t] = R^{-1}, \;\; E_t[(1+z_{D,t+\Delta t})\widehat{Y}_t(z_{D,t+\Delta t}) \big| S_t] = 1 \end{aligned} \;\; ,. \quad (4.4)$$

yielding the diffusion bounds $\underline{C}_{tD}(S_t, V_t)$ and $\overline{C}_{tD}(S_t, V_t)$ as in (3.6)-(3.8).

The final steps in this procedure are the incorporation of the jumps into (4.4) and the derivation of the limiting form of the bounds as the time partition $\Delta t \to 0$. The next theoretical result of this paper is as follows.

Proposition 3: For any $t < T$ and for $\Delta t \to 0$ the admissible stochastic dominance option values $C(S_t, V_t)$ for the SVJ model (4.1) lie in an interval of bounds $[\underline{C}(S_t, V_t), \bar{C}(S_t, V_t)]$ defined in their discretized version in (4.5) -(4.7), which converge to a pair of option price lower and upper bounds given by the expectation of the payoff with the following Q-dynamics.

$$\begin{aligned}
&\frac{dS_t}{S_t} = [r - (\lambda + \lambda_{Ut})\kappa_t^U]dt + \sigma(V)dW_1^Q + (j_t^U - 1)d\mathrm{N}, \text{ (upper bound)} \\
&\frac{dS_t}{S_t} = [r - \lambda\kappa_t^L]dt + \sigma(V)dW_1^Q + (j_t^L - 1)d\mathrm{N}, \text{ (lower bound)} \qquad , \\
&dV = \left(\alpha(V) - \frac{\rho(V)\beta(V)\gamma(V)}{\sigma(V)}\right)dt + \beta(V)[\rho(V)W_1^Q + \sqrt{1-\rho^2(V)}W_2^Q]
\end{aligned} \tag{4.5a}$$

$$\begin{aligned}
&j_t^U = \begin{cases} j & \text{with probability } \frac{\lambda}{\lambda + \lambda_{Ut}} \\ j_{\min} & \text{with probability } \frac{\lambda_{Ut}}{\lambda + \lambda_{Ut}} \end{cases}, \quad \lambda_{Ut}\Delta t = -\frac{\gamma(V_t)}{(j_{\min} - 1)}\Delta t \\
&E\left[j_t^U - 1 \right] = \kappa_t^U = (\frac{\lambda}{\lambda + \lambda_{Ut}})\kappa + (\frac{\lambda_{Ut}}{\lambda + \lambda_{Ut}})\left(j_{\min} - 1 \right) \qquad , \\
&\gamma(V_t) = \lambda\kappa - \lambda\kappa_t^L \\
&\lambda\kappa_t^L = E(j - 1 \mid j \le \bar{j}_t)
\end{aligned} \tag{4.5b}$$

Proof:

We prove the result for the upper bound, with the proof for the lower bound following identical steps. Given $C_{t+\Delta t}(S_{t+\Delta t}, V_{t+\Delta t}) \le \bar{C}_{t+\Delta t}^U(S_{t+\Delta t}, V_{t+\Delta t})$, we treat the P-distribution of $z_{t+\Delta t}$ as a mixture, with probabilities 1-$\lambda\Delta$t of a bivariate diffusion return $z_{D,t+\Delta t}$ and $\lambda\Delta t$ of the jump $(j-1)$. Hence, the problem (4.4) now becomes

$$\begin{aligned}
&Max_{\hat{Y}}\{E_t[\hat{Y}_t(z_{t+\Delta t})\bar{C}_{t+\Delta t}^U(S_{t+\Delta t}, V_{t+\Delta t})]\} = \\
&Max_{\hat{Y}}\{E_t\begin{bmatrix} (1-\lambda\Delta t)\hat{Y}_t(z_{D,t+\Delta t})\bar{C}_{t+\Delta t}^U(S_t(1+z_{D,t+\Delta t}), V_{t+\Delta t}) \\ +\lambda\Delta t\hat{Y}_t(z_{D,t+\Delta t})\bar{C}_{t+\Delta t}^U(S_t j), V_{t+\Delta t}) \end{bmatrix}\} = . \\
&Max_{\hat{Y}}\{E_t\begin{bmatrix} (1-\lambda\Delta t)\bar{C}_{tD}(S_t, V_t) \\ +\lambda\Delta t\hat{Y}_t(z_{D,t+\Delta t})\bar{C}_{t+\Delta t}^U(S_t j), V_{t+\Delta t}) \end{bmatrix}\}
\end{aligned} \tag{4.6}$$

A similar formulation holds for the lower bound, by replacing $\bar{C}_{t+\Delta t}^U$ by $\bar{C}_{t+\Delta t}^L$ and $\bar{C}_{tD}$ by $\underline{C}_{tD}$.

Since these bounds are by construction convex within the interval $(t, t+\Delta t]$, whenever a jump occurs the same argument used at $T-\Delta t$ about the jump components becoming dominant in the convolution holds here as well. It follows that the discrete time SVJ bounds become now expectations with the following index return dynamics for both upper and lower bounds, with the volatility updating common to both bounds

$$z_{t+\Delta t}^{U} = \begin{pmatrix} [r+\gamma(V_t)-(\lambda+\lambda_{Ut+\Delta t})\kappa_{t+\Delta t}^{U}]\Delta t+\sigma(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t} \text{ with prob. 1-}(\lambda+\lambda_{Ut+\Delta t})\Delta\text{t} \\ [r+\gamma(V_t)-(\lambda+\lambda_{Ut+\Delta t})\kappa_{t+\Delta t}^{U}]\Delta t+\sigma(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t}+(j_{t+\Delta t}^{U}-1) \text{ with prob. } (\lambda+\lambda_{Ut+\Delta t})\Delta t \end{pmatrix},$$
$$z_{t+\Delta t}^{L} = \begin{pmatrix} [r+\gamma(V_t)-\lambda\kappa_{t+\Delta t}^{L}]\Delta t+\sigma(V_t)\varepsilon_T\sqrt{\Delta t} \text{ with prob. 1-}\lambda\Delta\text{t} \\ [r+\gamma(V_t)-\lambda\kappa_{t+\Delta t}^{L}]\Delta t+\sigma(V_t)\varepsilon_T\sqrt{\Delta t}+(j_{t+\Delta t}^{L}-1) \text{ with prob. } \lambda\Delta t \end{pmatrix}.$$
$$\Delta V_t = [\alpha(V_t)-\frac{\rho(V_t)\beta(V_t)\gamma(V_t)}{\sigma(V_t)}]\Delta t+\beta(V_t)[\rho(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t}+\sqrt{1-\rho^2(V_t)}\eta_{t+\Delta t}\sqrt{\Delta t} \tag{4.7}$$

As at $T-\Delta t$, both intensity and amplitude of the jump component in the upper bound change at every time step if the risk premium is a function of volatility, while in the lower bound the intensity is unchanged from the P-dynamics but the amplitude is adjusted at every time step if the risk premium changes. Analytically,

$$j_{t+\Delta t}^{U} = \begin{cases} j & \text{with probability } \frac{\lambda}{\lambda+\lambda_{Ut+\Delta t}} \\ j_{\min} & \text{with probability } \frac{\lambda_{Ut+\Delta t}}{\lambda+\lambda_{Ut+\Delta t}} \end{cases},$$
$$E\left[j_{t+\Delta t}^{U}-1\right] = \kappa_{t+\Delta t}^{U} = (\frac{\lambda}{\lambda+\lambda_{Ut+\Delta t}})\kappa+(\frac{\lambda_{Ut+\Delta t}}{\lambda+\lambda_{Ut+\Delta t}})\left(j_{\min}-1\right) \tag{4.8a}$$

$$\gamma(V_t) = \lambda\kappa-\lambda\kappa_{t+\Delta t}^{L}$$
$$\lambda\kappa_{t+\Delta t}^{L} = E(j-1 \mid j \le \overline{j}_{t+\Delta t}). \tag{4.8b}$$

For $\Delta t \to 0$ the volatility term tends to the same limit (3.8) for both bounds, implying that the recursive bounds behave as in the constant volatility case and tend to (4.5ab), QED.

Proposition 3 provides endogenous risk neutralization expressions for an SVJ model, defined within two bounds for the corresponding option values. The SVJ model is extremely flexible, insofar as it allows all forms of the risk premium that have appeared in the financial literature, constant as well as dependent on variance or volatility in an unspecified way. Although the jump process was assumed to have constant intensity, it can be easily extended with very little reformulation to allow the dependence of intensity on volatility, as assumed in several studies. Note, however, that for such a model to be meaningful this dependence must be established on the basis of the P-distribution. In the Q-dynamics (4.8ab) the dependence holds anyway, given that the option value jump intensity for the upper bound $\lambda_{Ut+\Delta t}\Delta t = -\frac{\gamma(V_t)}{(j_{\min}-1)}\Delta t$ in (4.8b) is obviously dependent on the risk premium, which in turn may be dependent on current volatility.

As already noted, the inclusion of jumps in the evolution of volatility as reflected in the second equation of (4.1) needs a re-examination of the proof of Proposition 1, since at the continuous time limit the jump components, when they occur, will become dominant in the derivation of the risk neutral volatility. Since this would obviously raise serious estimation problems in filtering out the parameters of the jumps in volatility, it will be left for future empirical research.

Since the bounds (4.8ab) are based on SD, any efficient option price must lie within that interval. Otherwise, any risk averse investor holding the index and the riskless asset should purchase (write) the option if its price lies below (above) the interval. As for the determination of the efficient price within the SD bounds, this would depend on the aggregation of the pricing kernels of individual investors participating in the option market, whose risk aversion characteristics are unknown. The aggregation is also not necessarily constant within the maturity of the options. The last result of this paper describes these characteristics for the universe of CRRA investors who hold the index and the riskless asset and establishes a formal link between SD and NAE models.

Consider an investor who maximizes recursively the expectation of terminal wealth at option expiration time *T*, whose utility function is $E_t[\frac{W_T^{1-\varphi}}{1-\varphi}]$.[18] Maximizing $Max_{\alpha_{T-1}} E_{T-1}[\frac{W_T^{1-\varphi}}{1-\varphi}] = Max_{\alpha_{T-1}} E_{T-1}[W_{T-1}^{1-\varphi}\frac{\{\alpha_{T-1}(1+z_T)+(1-\alpha_{T-1})R\}^{1-\varphi}}{1-\varphi}]$, yielding the optimal allocation $\alpha_{T-1}^*(1+E_{T-1}(z_T))+(1-\alpha_{T-1}^*)R \equiv R_T^*$, defined from the first order conditions (FOC) up to $o(\Delta t)$, $\frac{E_{T-1}[R_T^{*-\varphi} z_T]}{E_{T-1}[R_T^{*-\varphi}]} = e^{r\Delta t}$, where we set the time partition equal to Δt. The Q-distribution is then $\frac{R_T^{*-\varphi} dF_{T-1}(z_T)}{E_{T-1}[R_T^{*-\varphi}]}$, with $F_{T-1}(z_T)$ denoting the distribution of the return in (4.2).

If the returns are independent and identically distributed (iid) then the allocation $\alpha_t^*(1+E_t(z_{t+\Delta t}))+(1-\alpha_t^*)R \equiv R_{t+\Delta t}^*$ remains unchanged over all time points. In SVJ, however, the returns $z_{t+\Delta t}$ are Markovian but not iid. In such a case the FOC at *t* become, when maximizing $E_t(W_{t+1})$ with respect to α_t, $\frac{E_t[(\prod_{\tau=t+1}^{\tau=T} R_\tau^*)^{-\varphi}(1+z_{t+1})]}{E_t[(\prod_{\tau=t+1}^{\tau=T} R_\tau^*)^{-\varphi}]} = e^{r\Delta t}$, since $W_t^* = W_{t+1}^* R_t^*$, and by a simple induction it can be shown that this recursive derivation is independent of wealth, as is common for such investors.

[18] The terminal wealth can extend to any time *T'*>*T* without changing the results below.

The corresponding kernel $\frac{(\prod_{\tau=t+1}^{\tau=T} R_\tau^*)^{-\varphi}}{E_t[(\prod_{\tau=t+1}^{\tau=T} R_\tau^*)^{-\varphi}]}$ is also equal to $\frac{S_T^{-\varphi}}{E_t[S_T^{-\varphi}]}$ and is dependent on the RRA parameter φ, which must be consistent with the SVJ bounds of Proposition 3. By definition, all risk averse investors no matter what their RRA will improve their expected utility if they adopt the appropriate policy that includes writing (purchasing) an overpriced (underpriced) option, as described below. Nonetheless, the correspondence between index holdings and adopted policy will not be one-to-one but will depend on the RRA. As with the constant volatility with jumps case in Ghanbari, Oancea and Perrakis (2021, p. 259),[19] the SD bounds set an upper limit on the RRA of the investors who participate in the option market with one option position per unit index if the options are priced "correctly". The SD lower bound and the Merton (1976) unsystematic jump risk case correspond to a risk neutral investor with an RRA of 0, so the RRA limit will come from the upper bound. This is formalized in the following result.

<u>Proposition 4</u>: Consider an admissible equilibrium value $C(S_t, V_t)$ of an option under SVJ P-dynamics as in (4.1), with $C(S_t, V_t) \in [\underline{C}(S_t, V_t), \bar{C}(S_t, V_t)]$ as in Proposition 2, as well as a set of CRRA investors defined by the size $\varphi \in (0, \infty)$ of their RRA. In such a case there is a time- and maturity-dependent maximum RRA value for such investors in order to participate in the option market with one option per unit index, given by

$$\varphi_{t,T}^{\max} = Max\left(\varphi \left| \begin{aligned} &\gamma(V_t) = \varphi V_t + \lambda\kappa - \lambda^Q \kappa^Q, \\ &\lambda^Q = \lambda E_t(j^{-\varphi}),\ \kappa^Q = \kappa E_t[\frac{(j-1)j^{-\varphi}}{E_t(j^{-\varphi})}] \end{aligned} \right. \right) \quad . \tag{4.9}$$

Conversely, for any $\varphi > \varphi_{t.T}^{max}$ the ratio $\frac{\varphi}{\varphi_{t,T}^{\max}}$ shows the number of index units per option position that the CRRA investor needs to hold if she is to participate in the option market. On the other hand, if there is an equilibrium corresponding to a "representative" CRRA investor within the bounds with $\varphi \in (0, \varphi_{t.T}^{max}]$, then the term within braces in the RHS of (4.9) shows the equilibrium price and the allocation of the premium to volatility and jump risk.

<u>Proof</u>: See Appendix C.

From (4.9) it is clear that the set $\varphi \in (0, \varphi_{t.T}^{max}]$ varies at every time point and is at the very least maturity-dependent even if we assume, as it often happens, that the volatility stays approximately constant for short maturities. This implies that the strong maturity effects noted

[19] In that paper, however, the RRA upper limit was extracted by equating the SD upper bound to the Bates (1991) model, in which the index risk premium was endogenously determined by simultaneous equilibrium in index and option markets. Since, as discussed in Perrakis (2022), that model has not had much success empirically, we shall not carry out the corresponding exercise.

earlier in the empirically fitted SVJ Q-distributions of the Bakshi, Cao and Chen (1997) study are perfectly reasonable in our SD setup. Further, that setup also provides useful benchmarks for investors who are not of the CRRA type and whose holdings may include other risky investments if their implied kernel is monotone decreasing in the index, a necessary condition for no arbitrage as discussed earlier.[20] Apart from that, our analysis did not impose any restrictions on the transition from P- to Q-dynamics, neither linearity of the volatility risk premium with respect to the variance as in Bates (1996, p. 74), nor restrictions on the Q-parameters as in Pan (2002, pp. 34-35).

Tables 4 and 5 show the derived SVJ option bounds for call options for SV following the SQRT model shown in the previous section, respectively for two different long run mean variances set equal to $V_0 = 0.0225$ and $V_0 = 0.01$, and equal to the corresponding starting volatilities of 15% and 10%. Each table shows the bounds for two different levels of the total risk premium, three different maturities of one month, three months and one year, and five degrees of moneyness of $\frac{K}{S} = 0.90, 0.95, 1, 1.05, 1.1$. The starting index level is 100 and the riskless rate 2%. The remaining parameters are broadly consistent with the parameter values of Bakshi, Cao and Chen (1997, Table 3), namely $\zeta = 1,\ \lambda = 0.6,\ \mu_j = E[\ln j \,|\, j \geq \underline{j}] = -0.05,\ \sigma_j^2 = Var[\ln j \,|\, j \geq \underline{j}] = (0.07)^2$ .The Monte Carlo approach was used for the results, with the Merton (1976) unsystematic risk neutral jump risk used as a control variate.

[Table 5 about here]

[Table 6 about here]

For the base case of a 4% risk premium our parameters imply $\kappa = -0.04877,\ \lambda_U = 0.2,\ \kappa^U = -0.08658 \Rightarrow (\lambda + \lambda_U)\kappa^U = -0.06926 = \lambda\kappa - 0.4$., and similarly for the economically less interesting lower bound we get $\kappa^L = 0.115437$.

The tables show that the SVJ lower bound is in all cases very close to the Merton (1976) risk neutral value, which is an alternative lower bound that does not truncate the amplitude. In both tables the results also exhibit a significant smile effect, especially in the ITM call region, which corresponds to OTM puts. For the base case of three-month options (T=0.25) the implied volatilities (IV) are 0.2 and 0.16 for starting diffusion volatilities of 15% and 10% respectively, rising to 0.23 and 0.21 for $\frac{K}{S_t} = 0.9$. These results are consistent with the claimed “overpriced OTM puts” anomalies in several NAE studies, as discussed in the introduction. There is no overpricing but the OTM puts have higher IV’s because of the jump components, as claimed by Broadie, Chernov and Johannes (2009).

[20] For a general risk averse investor the kernel depends on her wealth, which implies that for inference purposes it has to be standardized at one index unit.

The SVJ model is the "workhorse" of empirical option research, and the results of this section provide a template for empirical work that differs in major ways from previous studies. Such work starts from the estimation of the P-dynamics uniquely from the observed index returns, for which there is a long history of econometric work starting with Ait-Sahalia (2004). Once the P-parameters are available Proposition 3 yields the Q-dynamics, whose explicit form implies that any path to option maturity in the discretized from of the P-dynamics is mirrored by corresponding paths in the two Q-bounds. The latter can then be extracted by Monte Carlo or by a more refined numerical approach that transcends the scope of this paper.

The next step in the empirical work is the comparison of the SD bounds with the observed bid-ask spread in the option market. There are strong indications from the SD studies in the presence of frictions that the frictionless SD bounds will not be consistent with the option market. As Constantinides, Jackwerth and Perrakis (2009) first documented, most option cross sections are not consistent with a monotone decreasing pricing kernel, a sine qua non of the SD approach and, as argued above, also of the NAE approach. The inconsistency manifests itself with non-overlap or very little overlap of the frictionless SD bounds with the spread, which in many cases does not include the bid-ask midpoint, the universally used proxy for the frictionless option prices in NAE studies.

In the rest of this section we give a brief description of the design of strategies that may exploit this non-overlap, as well as of the formal empirical tests of the ex post profitability of these strategies, which are also tests of the consistency of the intermediated market with the frictionless equilibrium results. Since there is reason to believe that in many and perhaps most cross sections there are options with little or no overlap between the SD bounds and the bid-ask spread that lies above them, we focus on strategies to exploit the mispricing in such cases. Equivalent strategies exist for all other cases of inconsistency between the SD bounds and the option market. Such strategies are carried out in the frictionless economy at all time points of the index path till option maturity, which also assume that the OT can rebalance her portfolio between the index and riskless asset accounts without incurring any costs. The strategies can be defined for *any* IT portfolio holdings of index and riskless asset, with the ex post empirical tests simply adding to the IT holdings the dynamically adjusted zero net cost portfolio involving the overpriced option. In our illustrations below we shall concentrate on the case described in Proposition 4, where IT holds a single index unit and the trader-specific Q-dynamics are given by (4.10) with $\varphi \in (0, \varphi_{t.T}^{max}]$.

Suppose we observe call options in a number of cross sections of a given maturity, whose bid-ask spread lies above the SD upper bound, or $C_{bt} > \bar{C}(S_t, K, T)$. To exploit this mispricing OT adds at *t* to her index holdings of IT equal to S_t a portfolio of the proceeds from the short call, allocated optimally between the riskless asset and the index in respective proportions β_t and $1-\beta_t$ as described in Perrakis (2019, p. 25). The position is closed at $t+\Delta t$ at price $\bar{C}(S_{t+\Delta t}, K, T)$, with the upper bound derived from the trader-specific Q-dynamics (4.9)-(4.10).

At T this strategy yields an amount . $C_{bt}[\beta_t R^{T-t}+(1-\beta_t)\prod_{\tau=t}^{\tau=T-\Delta\tau}(1+z_{\tau+\Delta\tau})]-R^{T-t-\Delta t}\bar{C}(S_{t+\Delta t},K,T)$.

These OT proceeds are cumulated in a portfolio at every time τ for the set of times $\mathrm{T}=\{\tau:\ \tau\in[t,T],\ C_{b\tau}>\bar{C}(S_\tau,K,T)\}$ as the index moves along the discretized path of its dynamics (4.2), with the position closed at $\tau+\Delta\tau$ at the upper bound. $\bar{C}(S_{\tau+\Delta\tau},K,T)$. For each $\tau\in\mathrm{T}$ the allocation β_τ is chosen so that at the lowest value of the return $z_{\tau+\Delta\tau}$ the short option position at the upper bound $\bar{C}(S_{\tau+\Delta\tau},K,T)$ is equal to 0. This portfolio's composition is time- and state-dependent, and the quantity $C_{b\tau}\left[\beta_\tau R^{T-\tau}+(1-\beta_\tau)\prod_{\zeta=\tau}^{\zeta=T-\Delta\zeta}(1+z_{\zeta+\Delta\zeta})\right]-R^{T-\tau-\Delta\tau}\bar{C}(S_{\tau+\Delta\tau},K,T)$ is added at every point $\tau\in\mathrm{T}$ of the IT return path. Hence, the OT initial investment is again equal to S_t, while the final value is

$$S_T+\sum_{\tau\in\mathrm{T}}\left[C_{b\tau}[\beta_\tau R^{T-\tau}+(1-\beta_\tau)\prod_{\zeta=\tau}^{\zeta=T-\Delta\zeta}(1+z_{\zeta+\Delta\zeta})]-R^{T-\tau-\Delta\tau}\bar{C}(S_{\tau+\Delta\tau},K,T)\right]. \tag{4.11}$$

By construction, the time series of these OT returns in (4.11) in all cross sections in which the set $\tau\in\mathrm{T}$ is non empty and there are call options with $C_{b\tau}>\bar{C}(S_\tau,K,T)$ should stochastically dominate the index, a hypothesis that can be tested reliably by the Davidson-Duclos (2013) out-of-sample and model-free test. If the test rejects the null of non-dominance then the option market data is inconsistent with the estimated P -dynamics. At the very least, the corresponding options should be removed from the cross sections before any attempt to extract frictionless Q -dynamics from option market data. Such an extraction should only use option prices that lie within the SD bounds.

V. Summary and Conclusions

As noted in the introduction, the main objective of this paper is to extend the SD option pricing paradigm to the frictionless world, which is the one studied by the overwhelming majority of empirical index option market researchers. A key element of this extension is the derivation of SD bounds in the case of SV, when the diffusion volatility is stochastic, which had already been recognized as an important element of the index return dynamics more than thirty years ago. This element, however, made the frictionless derivative markets incomplete in the dominant NAE paradigm. In that paradigm the index return dynamics were insufficient to define the prices of derivative contracts without additional assumptions, several of which were highly questionable and generated long lasting and ongoing controversies.

As it turns out, the SD approach is by itself sufficient in order to derive a unique option price in continuous time, by extracting from the index returns an endogenous price of the volatility risk and the risk neutral return distribution. Proposition 1, the most important result of this paper, shows that under the SD assumptions the two recursively derived model free bounds that contain the efficient option price under SV dynamics converge to a single value even for the most general formulations of such dynamics that have appeared in earlier studies. If SV is then combined with independent jumps with random amplitude and constant intensity and becomes a stochastic volatility-jump process or SVJ the risk neutral transformations of SD result in two boundary distributions described in Proposition 3, as with constant volatility diffusion,. We then end up with two boundary values for the option that should contain the efficient equilibrium option prices for SVJ index return dynamics according to SD.

We argue that unique frictionless equilibrium option values within the bounds depend on the unobservable aggregation of trader preferences, which in turn depend on their asset holdings even for the simplest case of portfolios containing the index and a riskless bond. For the important special case of a CRRA trader we derive from the SD option upper bound a corresponding upper bound on the RRA in order to participate in the option market with one option position per index unit. We also derive for that CRRA trader the valuation of the volatility and jump risks when trading within the SD bounds.

We show how such a CRRA trader may exploit in the frictionless market an observed option price available for trading that lies outside the SD bounds. For these cases we derive dynamic strategies tailored to a particular CRRA investor that allocate efficiently the proceeds from the mispriced option to the index and the riskless asset account. Most importantly, we also show how to subject such strategies to ex post out-of-sample profitability tests using only the observable path of the index to option expiration. If these tests show that the returns from the mispriced option when added to the index create a stochastically dominant position vis-à-vis the index holdings then we may conclude that the observed option market prices are inconsistent with the index dynamics and cannot be used in frictionless option pricing models.

The frictionless market strategies can also be applied to a market with frictions, although in such a case closing an open position requires the use of the appropriate bid or ask price, rather than the frictionless bound. Alternatively, the position can be left open till maturity, as in Constantinides *et al* (2011). More interesting empirically, however, is the analysis of the intermediate market and an interpretation of the disconnection between the estimated *P*-dynamics and the observed prices in the option market whenever a non-overlap between SD bounds and the bid-ask spread. In such cases the SVJ bounds on efficient frictionless option prices may also be used in order to evaluate the positions of the market makers or dealers in the intermediate market. These dealers provide the residual liquidity in order to close the market for each traded option. The liquidity file data that allocates the positions to dealers and non-dealer end users is available and allows us to value the net exposure of the dealers within the SD bounds. Adding to this net exposure value the riskless proceeds of the dealer group as a whole from intermediating the end user transactions is certainly a useful exercise in order to assess the perfect competition assumption of the dealer market. Such an assumption has never been tested but should be a priori suspicious in view of the informational asymmetry between end user traders and market makers.

Appendix A: Proof of Lemma 1

The proof of Lemma 1, which was initially given by Ritchken (1985), will only be sketched, since it is also given in Perrakis (2019, pp. 30-36). It is based on a linear programming (LP) formulation of the equilibrium conditions (3.4), which relies on the monotonicity of the kernel $\widehat{Y}_t(z_{t+\Delta t})$. As in (3.5), define $\hat{z}_i = E_t[z(\varepsilon_j)\,|\, j \le i],\ \ i = 1,...,n,$ and similarly define

$$\hat{\underline{C}}_{i,t+\Delta t} = E_t[\underline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon,\eta))\,|\, z(\varepsilon) \le \hat{z}_i],\ \ \hat{\bar{C}}_{i,t+\Delta t} = E_t[\bar{C}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon,\eta))\,|\, z(\varepsilon) \le \hat{z}_i] \quad \text{(A.1)}$$

Let also $\Upsilon_1 \equiv \widehat{Y}_t(z_{t+\Delta t}(\varepsilon_1)), \Upsilon_2 = \widehat{Y}_t(z_{t+\Delta t}(\varepsilon_1)) - \widehat{Y}_t(z_{t+\Delta t}(\varepsilon_2)),...., \Upsilon_n = \widehat{Y}_t(z_{t+\Delta t}(\varepsilon_{n-1})) - \widehat{Y}_t(z_{t+\Delta t}(\varepsilon_n))$. The problem (3.4) then takes the following form, with the dependence on η suppressed for notational simplicity.

$$\begin{aligned} & max_{\Upsilon_i} \sum_{i=1}^{i=n} \Upsilon_i \hat{\bar{C}}_{i,t+\Delta t} (\min_{\Upsilon_i} \sum_{j=1}^{j=n} \Upsilon_i \hat{\underline{C}}_{i,t+\Delta t})] \\ & subject\ to \\ & \sum_{j=1}^{j=n} \bar{Y}_j = R^{-1},\ \sum_{i=1}^{i=n} \Upsilon_i (1+\hat{z}_{i,t+\Delta t}) = 1,\ \Upsilon_i \ge 0,\ i = 1,...,n \end{aligned} \quad . \quad \text{(A.2)}$$

The solution yields (3.6) and (3.7), from which (3.5) follows obviously, since $E_t^L[\underline{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon,\eta))] \le C(S_t, V_t\,|\,\eta) \le E_t^U[\bar{C}_{t+\Delta t}(S_{t+\Delta t}(\varepsilon), V_{t+\Delta t}(\varepsilon,\eta))]$, QED.

Appendix B: Proof of Proposition 2

At time *t* and for the period Δt we have, from the proof of Proposition 1,

$$\hat{Y}(z^P_{t+\Delta t}) = \frac{\hat{q}_t(\varepsilon)}{\hat{p}_t(\varepsilon)} = \frac{\Pr ob[\frac{\Delta S_t}{S_t} = z^Q_{t+\Delta t},\ \Delta V_t = \hat{\alpha}(V_t)\Delta t + \beta(V_t)\rho(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t}]}{\Pr ob[\frac{\Delta S_t}{S_t} = z^P_{t+\Delta t},\ \Delta V_t = \alpha(V_t)\Delta t + \beta(V_t)\rho(V_t)\varepsilon_{t+\Delta t}\sqrt{\Delta t}]}. \quad \text{(B.1)}$$

At the limit, setting $z^Q_{t+\Delta t} = \ln[\frac{\Delta S_t}{S_t}]^Q,\ \ z^P_{t+\Delta t} = \ln[\frac{\Delta S_t}{S_t}]^P$, we have in approximating their convergence values,

$$\begin{aligned} & z^Q_{t+\Delta t} = \ln[\frac{\Delta S_t}{S_t}]^Q \sim N(r - 0.5V, \sigma(V_t)),\ \ z^P_{t+\Delta t} = \ln[\frac{\Delta S_t}{S_t}]^P \sim N(r + \gamma(V_t) - 0.5V, \sigma(V_t)) \\ & \Delta V_t^Q \sim N(\hat{\alpha}(V_t), \beta(V_t)\rho(V_t)),\ \ \Delta V_t^P \sim N(\alpha(V_t), \beta(V_t)\rho(V_t)) \end{aligned} \quad . \quad \text{(B.2)}$$

Since both *P*- and *Q*- distributions depend on a single random variable $z^P_{t+\Delta t}$ or $z^Q_{t+\Delta t}$, the probabilities are normal and must be equal for the return and volatility dynamics. It follows, therefore, that

$$\begin{aligned}&\frac{1}{\sigma(V_t)}\exp\{-\frac{(r-0.5\sigma^2(V_t)-z^P)^2}{2\sigma^2(V_t)}\}=\frac{1}{-\beta(V_t)\rho(V_t)}\exp\{-\frac{(\alpha(V_t)-z^P)^2}{2\beta^2(V_t)\rho^2(V_t)}\}\\&\frac{1}{\sigma(V_t)}\exp\{-\frac{(r+\gamma(V_t)-0.5\sigma^2(V_t)-z^Q)^2}{2\sigma^2(V_t)}\}=\frac{1}{-\beta(V_t)\rho(V_t)}\exp\{-\frac{(\hat{\alpha}(V_t)-z^Q)^2}{2\beta^2(V_t)\rho^2(V_t)}\}\end{aligned}. \tag{B.3}$$

Standardizing them, we define

$$\begin{aligned}&z_s^P=\frac{z^P}{\sigma(V_t)},\ \ z_v^P=\frac{z^P}{|\beta(V_t)\rho(V_t)|},\ \ \mu_s^P=\frac{r+\gamma(V_t)-0.5\sigma^2(V_t)}{\sigma(V_t)},\ \ \mu_v^P=\frac{\alpha(V_t)}{|\beta(V_t)\rho(V_t)|}\\&z_s^Q=\frac{z^Q}{\sigma(V_t)},\ \ z_v^Q=\frac{z^Q}{|\beta(V_t)\rho(V_t)|},\ \ \mu_s^Q=\frac{r+\gamma(V_t)-0.5\sigma^2(V_t)}{\sigma(V_t)},\ \ \mu_v^Q=\frac{\hat{\alpha}(V_t)}{|\beta(V_t)\rho(V_t)|}\end{aligned}. \tag{B.4}$$

In turn, these yield

$$\exp(-\frac{(\mu_s^P-z_s^P)^2}{2})=\exp(-\frac{(\mu_v^P-z_v^P)^2}{2}),\quad \exp(-\frac{(\mu_s^Q-z_s^Q)^2}{2})=\exp(-\frac{(\mu_v^Q-z_v^Q)^2}{2}). \tag{B.5}$$

It follows then immediately that $z_s^P-z_v^P=z^P[\frac{1}{\sigma(V_t)}-\frac{1}{|\beta(V_t)\rho(V_t)|}]\equiv\frac{z^P}{\hat{\sigma}}$, where

$\hat{\sigma}=\left|\frac{1}{\sigma(V_t)}-\frac{1}{|\beta(V_t)\rho(V_t)|}\right|^{-1}$. Assume $\sigma(V_t)>|\beta(V_t)\rho(V_t)|$ and $\rho(V_t)<0$, in which case $\hat{\sigma}=\frac{|\beta(V_t)\rho(V_t)\sigma(V_t)|}{\sigma(V_t)-\beta(V_t)\rho(V_t)}$. Then we clearly have

$$\begin{aligned}&(\mu_s^P-z_s^P)=(\mu_v^P-z_v^P)\Leftrightarrow\mu_s^P-\mu_v^P=\frac{z^P}{\hat{\sigma}}\ \text{ for every } z^P\\&(\mu_s^Q-z_s^Q)=(\mu_v^Q-z_v^Q)\Leftrightarrow\mu_s^Q-\mu_v^Q=\frac{z^Q}{\hat{\sigma}}\ \text{ for every } z^Q\end{aligned}. \tag{B.6}$$

It can be easily seen that $\mu^P=\mu^Q$, and from (B.1) we have

$$\begin{aligned}\hat{Y}(z_{t+\Delta t}^P)&=\frac{n[\mu^Q,\hat{\sigma}]}{n[\mu^P,\hat{\sigma}]}=\frac{\exp[-\frac{1}{2}(\mu^Q-\hat{z}^Q)^2]}{\exp[-\frac{1}{2}(\mu^P-\hat{z}^P)^2]}=\exp[\frac{1}{2}(\mu^P-\mu^Q-\hat{z}^P+\hat{z}^Q)(\mu^P+\mu^Q-\hat{z}^P-\hat{z}^Q)]\\&=\exp[\frac{1}{2}(\frac{\gamma(V_t)}{\hat{\sigma}})(2\mu^P-\frac{\gamma(V_t)}{\hat{\sigma}}-2\hat{z}^P)]=\exp[\frac{1}{2}(\frac{\gamma(V_t)}{\hat{\sigma}})(2\mu^Q+\frac{\gamma(V_t)}{\hat{\sigma}}-2\hat{z}^Q)]\end{aligned}, \tag{B.7}$$

QED.

Appendix C: Proof of Proposition 4

The kernel $\frac{S_T^{-\varphi}}{E_t[S_T^{-\varphi}]}$ is obviously monotone decreasing in S_T, from which it follows that the implied boundary Q-distributions are given by (2.4). In such a case the upper bound is the same as the constant volatility jump diffusion, in which the key Θ-probability is equal to $-\frac{\gamma(V_t)}{(j_{\min}-1)}\Delta t \equiv \lambda_{Ut}\Delta t$. For (4.9) we set by changing the notation $\frac{S_{t+\Delta t}}{S_t} = \exp(z_{D,t+\Delta t} + J)$. Marginal analysis of borrowing $1 and investing in the index should in equilibrium yield $\begin{pmatrix} E_t[S_{t+\Delta t}^{-\varphi}[\exp(z_{D,t+\Delta t} + J - r)]] = 0 \Rightarrow \\ E_t\{\exp[-\varphi(z_{D,t+\Delta t} + J)]\exp(z_{D,t+\Delta t} + J)\} = rE_t\{\exp[-\varphi(z_{D,t+\Delta t} + J)]\} \end{pmatrix}$. Since the two random terms are independent, the second line becomes

$$E_t\{\exp[(1-\varphi)z_{D,t+\Delta t}]\}E_t\{\exp[(1-\varphi)J)]\} = rE_t\{\exp(-\varphi z_{D,t+\Delta t})E_t\{\exp(-\varphi J)\}$$

from which we get (4.9) after setting $\exp(-\varphi J) = j^{-\varphi}$, QED.

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Table 1: SV-SQRT

The table shows the option values corresponding to a $100 price of the S&P 100 index for the shown values of volatility and maturity in days under SD and NAE, with strike prices equal to the ones shown on the top line; only OTM values are shown, puts on the right and calls on the left. The SV model follows the SQRT format in both cases, with the *P*-parameters corresponding to those shown in Table 1 of Jones (2003, p. 197).

			80	90	95	99	100	101	105	110	120
Pricing	T	V_0									
NAE	5	0.0010	0.0031	0.2392	1.0036	2.3866	2.8664	2.4015	1.0629	0.2966	0.0093
		0.0001	-0.0000	0.0001	0.0221	0.5266	0.9276	0.4974	0.0067	0.0000	0
	22	0.0010	0.6273	2.4116	4.0738	5.8445	6.3500	5.8805	4.2451	2.7076	0.9550
		0.0001	0.0022	0.1318	0.6344	1.7618	2.1995	1.7091	0.4616	0.0391	0.0000
	66	0.0010	4.4088	7.9112	10.1449	12.1620	12.6976	12.2455	10.5577	8.7077	5.7872
		0.0001	0.4204	1.6747	2.9908	4.5239	4.9831	4.4742	2.7641	1.3361	0.2085
SD	5	0.0010	0.0021	0.2059	0.9269	2.2859	2.7641	2.2997	0.9806	0.2551	0.0064
		0.0001	-0.0000	0.0000	0.0182	0.4987	0.8967	0.4691	0.0049	0.0000	0
	22	0.0010	0.3372	1.7308	3.2301	4.9303	5.4284	4.9557	3.3491	1.9256	0.5109
		0.0001	0.0006	0.0706	0.4503	1.4810	1.9114	1.4246	0.2883	0.0116	0.0000
	66	0.0010	1.5282	3.9234	5.7771	7.6079	8.1145	7.6406	5.9357	4.2140	1.9365
		0.0001	0.0819	0.6776	1.6204	2.9766	3.4217	2.9132	1.3553	0.3754	0.0091

Table 2: SV-CEV

The table shows the option values corresponding to a $100 price of the S&P 100 index for the shown values of volatility and maturity in days under SD and NAE, with strike prices equal to the ones shown on the top line; only OTM values are shown, puts on the right and calls on the left. The SV model follows the CEV format in both cases, with the *P*-parameters corresponding to those shown in Table 1 of Jones (2003, p. 197).

			80	**90**	**95**	**99**	**100**	**101**	**105**	**110**	**120**
Pricing	T	V_0									
NAE	5	0.0010	0.0177	0.3484	1.1250	2.4467	2.9040	2.4174	1.0108	0.2339	0.0027
		0.0001	0.0000	0.0001	0.0231	0.5290	0.9314	0.5019	0.0074	0.0000	0.0000
	22	0.0010	1.2649	3.1043	4.6443	6.2584	6.7192	6.2049	4.3912	2.6640	0.7619
		0.0001	0.0075	0.1722	0.6958	1.8233	2.2604	1.7654	0.4920	0.0430	0.0000
	66	0.0010	7.0404	10.2110	12.1612	13.9142	14.3917	13.8710	11.9030	9.7032	6.1833
		0.0001	0.9874	2.4503	3.7795	5.2594	5.6939	5.1595	3.3225	1.7094	0.3064
SD	5	0.0010	0.0136	0.2988	1.0366	2.3383	2.7973	2.3114	0.9267	0.1971	0.0017
		0.0001	0.0000	0.0000	0.0180	0.4994	0.8982	0.4721	0.0056	0.0000	0.0000
	22	0.0010	0.7621	2.2612	3.6675	5.2295	5.6859	5.1717	3.4064	1.8330	0.3635
		0.0001	0.0016	0.0851	0.4710	1.4977	1.9273	1.4401	0.2988	0.0122	0.0000
	66	0.0010	2.9989	5.3451	7.0148	8.6407	9.0867	8.5548	6.6101	4.6035	1.9438
		0.0001	0.1710	0.8350	1.7637	3.0759	3.5089	2.9881	1.3868	0.3774	0.0077

Table 3: SV-2GAM

The table shows the option values corresponding to a \$100 price of the S&P 100 index for the shown values of volatility and maturity in days under SD and NAE, with strike prices equal to the ones shown on the top line; only OTM values are shown, puts on the right and calls on the left. The SV model follows the 2GAM format in both cases, with the *P*-parameters corresponding to those shown in Table 1 of Jones (2003, p. 197).

			80	**90**	**95**	**99**	**100**	**101**	**105**	**110**	**120**
Pricing	*T*	V_0									
NAE	5	0.0010	0.0450	0.4261	1.1706	2.4092	2.8425	2.3327	0.8719	0.1401	0.0001
		0.0001	0.0000	0.0002	0.0259	0.5354	0.9347	0.5028	0.0071	0.0000	0.0000
	22	0.0010	1.2659	2.8015	4.1262	5.5678	5.9773	5.4259	3.5031	1.7617	0.2231
		0.0001	0.0153	0.2123	0.7493	1.8614	2.2878	1.7855	0.4892	0.0387	0.0000
	66	0.0010	3.9155	6.1937	7.7783	9.3130	9.7296	9.1721	7.1177	4.9606	2.0084
		0.0001	1.0704	2.4376	3.6834	5.0892	5.5051	4.9551	3.0681	1.4409	0.1591
SD	5	0.0010	0.0338	0.3634	1.0703	2.2900	2.7246	2.2165	0.7872	0.1119	0.0000
		0.0001	0.0000	0.0001	0.0199	0.4998	0.8954	0.4676	0.0051	0.0000	0.0000
	22	0.0010	0.8094	2.0727	3.2879	4.6889	5.1130	4.5656	2.7081	1.1583	0.0766
		0.0001	0.0028	0.0919	0.4703	1.4778	1.9005	1.4102	0.2775	0.0099	0.0000
	66	0.0010	2.0267	3.7946	5.1865	6.6340	7.0488	6.4881	4.4845	2.5534	0.5219
		0.0001	0.1768	0.7837	1.6584	2.9354	3.3642	2.8413	1.2590	0.3110	0.0038

Table 4: SVJ for V_0 =0.0225

The table shows the call option values under the indicated conditions, for parameters ζ=1, λ=0.6, log-amplitude mean and volatility equal to -0.05 and 0.07 respectively, and for S=100 and r=0.02.

Risk premium	K	T	Risk Neutral (Closed Form)	Lower Bound	Risk Neutral	Upper Bound	Upper Bound (Jmin=0)
0.04	90	1M	10.2135	10.2122	10.2182	10.3873	10.5068
		3M	10.8702	10.8610	10.8766	11.2888	11.6815
		1Y	13.9123	13.8624	13.8941	14.9039	16.6697
	95	1M	5.5104	5.4963	5.5114	5.7109	5.7886
		3M	6.7024	6.6787	6.7076	7.1544	7.4320
		1Y	10.4093	10.3365	10.3905	11.5015	12.9711
	100	1M	1.9142	1.8910	1.9161	2.0573	2.0935
		3M	3.4649	3.4218	3.4634	3.8472	4.0169
		1Y	7.4533	7.3663	7.4334	8.5613	9.7229
	105	1M	0.3303	0.3113	0.3288	0.3736	0.3835
		3M	1.4291	1.3836	1.4277	1.6696	1.7556
		1Y	5.0862	4.9891	5.0723	6.1222	6.9921
	110	1M	0.0268	0.0201	0.0261	0.0332	0.0328
		3M	0.4545	0.4211	0.4562	0.5650	0.5976
		1Y	3.2976	3.2069	3.2939	4.1814	4.8071
0.06	90	1M	10.2135	10.2033	10.2182	10.4650	10.6533
		3M	10.8702	10.8297	10.8766	11.4871	12.0893
		1Y	13.9123	13.7806	13.8941	15.3549	18.0799
	95	1M	5.5104	5.4693	5.5114	5.8018	5.9288
		3M	6.7024	6.6256	6.7076	7.3719	7.8056
		1Y	10.4093	10.2364	10.3905	11.9935	14.3143
	100	1M	1.9142	1.8618	1.9161	2.1244	2.1866
		3M	3.4649	3.3636	3.4634	4.0336	4.3088
		1Y	7.4533	7.2558	7.4334	9.0614	10.9504
	105	1M	0.3303	0.3010	0.3288	0.3966	0.4125
		3M	1.4291	1.3423	1.4277	1.7915	1.9363
		1Y	5.0862	4.8792	5.0723	6.5964	8.0620
		1M	0.0268	0.0192	0.0261	0.0342	0.0362
		3M	0.4545	0.4008	0.4562	0.6192	0.6813
	110	1Y	3.2976	3.1044	3.2939	4.6046	5.6919
		3M	0.0863	0.0635	0.0908	0.1068	0.1092
		1Y	1.6310	1.5621	1.6375	2.0927	2.2476

Table 5: SVJ for V_0 =0.01

The table shows the call option values under the indicated conditions, for parameters ζ=1, λ=0.6, log-amplitude mean and volatility equal to -0.05 and 0.07 respectively, and for S=100 and r=0.02.

Risk premium	K	T	Risk Neutral (Closed Form)	Lower Bound	Risk Neutral	Upper Bound	Upper Bound (Jmin=0)
0.04	90	1M	10.1931	10.1924	10.1968	10.3702	10.4889
		3M	10.6378	10.6351	10.6454	11.0962	11.4904
		1Y	12.8637	12.8229	12.8494	14.0651	15.8816
	95	1M	5.3267	5.3148	5.3279	5.5454	5.6258
		3M	6.1257	6.1011	6.1306	6.6535	6.9323
		1Y	8.9851	8.9119	8.9718	10.3752	11.8248
	100	1M	1.3602	1.3300	1.3618	1.5153	1.5498
		3M	2.5466	2.4904	2.5465	2.9871	3.1426
		1Y	5.7380	5.6301	5.7245	7.1469	8.2249
	105	1M	0.0759	0.0577	0.0753	0.0949	0.0974
		3M	0.6249	0.5692	0.6267	0.8273	0.8826
		1Y	3.2716	3.1432	3.2660	4.5075	5.2427
	110	1M	0.0051	0.0004	0.0049	0.0055	0.0058
		3M	0.0863	0.0592	0.0908	0.1282	0.1376
		1Y	1.6310	1.5112	1.6375	2.5449	2.9981
0.06	90	1M	10.1931	10.1872	10.1968	10.4491	10.6364
		3M	10.6378	10.6096	10.6454	11.3132	11.9164
		1Y	12.8637	12.7377	12.8494	14.5743	17.4077
	95	1M	5.3267	5.2831	5.3279	5.6449	5.7758
		3M	6.1257	6.0345	6.1306	6.9109	7.3428
		1Y	8.9851	8.7808	8.9718	10.9841	13.3063
	100	1M	1.3602	1.2933	1.3618	1.5902	1.6496
		3M	2.5466	2.4075	2.5465	3.2076	3.4627
		1Y	5.7380	5.4664	5.7245	7.7887	9.5853
	105	1M	0.0759	0.0536	0.0753	0.1035	0.1102
		3M	0.6249	0.5280	0.6267	0.9355	1.0367
		1Y	3.2716	2.9746	3.2660	5.1028	6.3984
	110	1M	0.0051	0.0003	0.0049	0.0063	0.0061
		3M	0.0863	0.0517	0.0908	0.1505	0.1728
		1Y	1.6310	1.3740	1.6375	3.0311	3.8810